THREE NONRELATIVISTIC QUARKS IN THE LATTICE QCD POTENTIAL: CAN ONE SEE THE DIFFERENCE IN BARYON SPECTRA?*

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We used the $\mathrm{U}(1)\otimes\mathrm{SO}(3)_{\mathrm{rot}} \subset \mathrm{U}(3) \subset \mathrm{SO}(6)$ hyperspherical harmonics of I. Salom, V. Dmitrašinović, *Nucl. Phys. B* **920**, 521 (2017). to calculate the energy-spectrum of three nonrelativistic quarks in the (interpolation of the) lattice QCD potential. We show that the first clear difference between the Δ, or the Y-string confinement and the lattice QCD potential can be seen only in the third shell of excited states. This is beyond experimental access, even in the light-quark sector. We also briefly discuss the role of relativity.

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1. Introduction

The form of the three-heavy-quark potential in (lattice) QCD is substantially better known after the recent lattice work by Sakumichi and Suganuma [1], and by Koma and Koma [2]. The form of this potential has been analysed in terms of hyperspherical variables by Leech *et al.* [3, 4], as well as in another contribution to this workshop [5], where an upper and a lower bound (a band of small width) on the triangle-shape dependence of the confining part of the potential has been established.

These upper and lower bounds on the potential in two sectors/lines in the shape space can be extrapolated to the whole shape space due to its periodicity, and certain inequalities can be inferred about the value of the v_{66} hyperspherical expansion coefficient. In this light, one may even discuss the consequences of this lattice potential in three-heavy-quark spectroscopy. The aim of this work is to briefly discuss the effects of the lattice QCD 3-quark potential, as extracted in Refs. [3–5], in the heavy-baryon spectrum, as calculated in the hyperspherical approach, see Refs. [6–9].

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2. The nonrelativistic quantum-mechanical three-body problem in hyperspherical coordinates

2.1. O(6) hyperspherical coordinates

Any (spin-independent) three-body potential must be invariant under: (1) translations; (2) overall ("ordinary $O(3)$") rotations; (3) permutations, if three identical particles are involved. Thus, due to (1), it may depend only on the relative position (Jacobi) vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\boldsymbol{x_1} - \boldsymbol{x_2})$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\boldsymbol{x_1} + \boldsymbol{x_2} - 2\boldsymbol{x_3})$; (2) is a function of three scalar products of the two vectors, $\boldsymbol{\rho}\cdot\boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$; (3) it must be permutation-symmetric.

It can be transcribed in hyperspherical coordinates as $f(R, \Omega_5)$, where $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the hyperradius, and five angles Ω_5 that parametrize a hypersphere in the six-dimensional Euclidean space. Three (Φ_i; $i = 1, 2, 3$) of these five angles (Ω_5) are just the Euler angles associated with the orientation in a three-dimensional space of a spatial reference frame defined by the (plane of) three bodies; the remaining two hyperangles describe the shape of the triangle subtended by three bodies; they are functions of three independent scalar three-body variables, *e.g.*, $\boldsymbol{\rho}\cdot\boldsymbol{\lambda}$, $\boldsymbol{\rho}^2$, and $\boldsymbol{\lambda}^2$. One linear combination ($\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2$) of the two variables is already taken by the hyperradius R, so the shape-space is two-dimensional, and topologically equivalent to the surface of a three-dimensional sphere. We define the hyperangles (α, ϕ) as $(\sin\alpha)^2 = 1 - \left(\frac{2\boldsymbol{\rho}\times\boldsymbol{\lambda}}{R^2}\right)^2$, $\tan\phi = \left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right)$, which reveal the full S_3 permutation symmetry of the problem: the angle α does not change under permutations, so that all permutation properties are encoded in the ϕ-dependence of the wave functions. This leads to permutation-adapted hyperspherical harmonics, as explained in Refs. [6, 7] wherein specific hyperspherical harmonics used here are displayed.

2.2. O(6) harmonics

Labelling the $O(6)$ hyperspherical harmonics with labels K, the Abelian hyperangular momentum quantum number Q conjugated with the Iwai angle ϕ, the (total orbital) angular momentum quantum numbers L and $L_z = m$, and ν which is the multiplicity label that distinguishes between hyperspherical harmonics with remaining four quantum numbers that are identical, as defined in Refs. [6, 7], corresponds to the subgroup chain $\mathrm{U}(1) \otimes \mathrm{SO}(3)_{\mathrm{rot}} \subset \mathrm{U}(3) \subset \mathrm{SO}(6)$. We expand the wave function $\Psi(R, \Omega_5)$ in terms of hyperspherical harmonics $\mathcal{Y}^K_{[m]}(\Omega_5)$, $\Psi(R, \Omega_5) = \sum_{K,[m]} \psi^K_{[m]}(R)\mathcal{Y}^K_{[m]}(\Omega_5)$.

2.3. Hyperspherical expansion of three-body Schrödinger equation

The hyperspherical harmonics turn the Schrödinger equation of three particles in a factorizable three-body potential $V(R,\alpha,\phi) = V(R)V(\alpha,\phi)$ into a set of coupled hyperradial equations

$$\begin{aligned} &-\frac{1}{2\mu}\left[\frac{\mathrm{d}^2}{\mathrm{d}R^2} + \frac{5}{R}\frac{\mathrm{d}}{\mathrm{d}R} - \frac{K(K+4)}{R^2} + 2\mu E\right]\psi^K_{[m]}(R) \\ &+ V_{\mathrm{eff}}(R)\sum_{K',[m']} C^{K\ \ K'}_{[m][m']}\psi^{K'}_{[m']}(R) = 0 \end{aligned} \tag{1}$$

with a hyperangular coupling coefficients matrix $C^{K\ \ K'}_{[m][m']}$ defined by

$$\begin{aligned} V_{\mathrm{eff}}(R)C^{K'\ \ K}_{[m'][m]} &= \left\langle \mathcal{Y}^{K'}_{[m']}(\Omega_5)\right| V(R,\alpha,\phi)\left|\mathcal{Y}^{K}_{[m]}(\Omega_5)\right\rangle \\ &= V(R)\left\langle \mathcal{Y}^{K'}_{[m']}(\Omega_5)\right| V(\alpha,\phi)\left|\mathcal{Y}^{K}_{[m]}(\Omega_5)\right\rangle . \end{aligned} \tag{2}$$

Factorizability of the potential is a simplifying assumption that leads to analytic results in the energy spectrum. It holds for the power-law ones, but also other homogeneous ones.

2.4. Hyperspherical expansion of three-body potentials

The hyperangular part $V(\alpha,\phi)$ of a factorizable potential can be expanded in terms of $O(6)$ hyperspherical harmonics with zero angular momenta $L = m = 0$ as

$$V(\alpha,\phi) = \sum_{K,Q}^{\infty} v^{3\text{-body}}_{K,Q}\mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi)\,, \tag{3}$$

where

$$v^{3\text{-body}}_{K,Q} = \int \mathcal{Y}^{KQ\nu*}_{00}(\Omega_5)\,V(\alpha,\phi)\,\mathrm{d}\Omega_{(5)} \tag{4}$$

leading to

$$\begin{aligned} V_{\mathrm{eff}}(R)C^{K''\ \ K'}_{[m''][m']} &= V(R)\sum_{K,Q}^{\infty} v^{3\text{-body}}_{K,Q} \\ &\times\left\langle \mathcal{Y}^{K''}_{[m'']}(\Omega_5)\right| \mathcal{Y}^{KQ\nu}_{00}(\alpha,\phi)\left|\mathcal{Y}^{K'}_{[m']}(\Omega_5)\right\rangle . \end{aligned} \tag{5}$$

In the case of three identical particles, the sum runs only over double-even-order ($K = 0, 4, \ldots$) $O(6)$ hyperspherical harmonics with zero value of the democracy quantum number $G_3 = Q = 0$, as well as over $K = 6, 12, 18\ldots$ $O(6)$ hyperspherical harmonics with democracy quantum number $G_3 \equiv Q \equiv 0 \pmod 6$, always with vanishing angular momentum $L = m = 0$.

The numerical values for the first four allowed (nonvanishing) $v_{K,Q}^{\text{3-body}}$ coefficients for $K \leq 11$, in the Y- and Δ-string and Coulomb potential's hyperspherical expansions are tabulated in Table I. All other coefficients vanish for $K < 12$. Smallness of the coefficient $v_{6,\pm 6}$(Y-string) indicates (an additional) dynamical symmetry of the Y-string potential.

TABLE I

Expansion coefficients v_{KQ} of the Y- and Δ-string as well as of the Coulomb and Logarithmic potentials in terms of $O(6)$ hyperspherical harmonics $\mathcal{Y}_{0,0}^{K,0,0}$, for $K =$ 0, 4, 8, respectively, and of the hyperspherical harmonics $\mathcal{Y}_{0,0}^{6,\pm 6,0}$.

(K,Q)	v_{KQ}(Y-string)	v_{KQ}(Δ-string)	v_{KQ}(CM-string)	v_{KQ}(Coulomb)
(0,0)	8.22	16.04	$16.04/\sqrt{3}$	20.04
(4,0)	−0.398	−0.445	$-0.445/\sqrt{3}$	2.93
$(6,\pm 6)$	−0.027	−0.14	$0.14/\sqrt{3}$	1.88
(8,0)	−0.064	−0.04	$-0.04/\sqrt{3}$	1.41

3. Consequences for the low-lying spectrum

The lowest-lying states in which the energy splittings depend on the v_{66} coefficient are the odd-parity resonances in the $K = 3$ shell. In Ref. [8], one finds that only two pairs of levels in the $K = 3$ shell are split by the v_{66} coefficient

$$
\begin{aligned}
&[20, 1^-]\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{\sqrt{3}}v_{40} - \frac{2}{7}v_{66}\right),\\
&[56, 1^-]\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00} + \frac{1}{\sqrt{3}}v_{40} + \frac{2}{7}v_{66}\right),\\
&[20, 3^-]\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{\sqrt{3}}{7}v_{40} - v_{66}\right),\\
&[56, 3^-]\ \frac{1}{\pi\sqrt{\pi}}\left(v_{00} - \frac{\sqrt{3}}{7}v_{40} + v_{66}\right).
\end{aligned}
\tag{6}
$$

In Table I, one can find the hyperspherical harmonic expansion coefficients of several standard potentials. In a separate contribution to this conference [5], we have analysed the lattice QCD data from Refs. [1, 2] in terms of permutation-adapted hyperspherical variables. We found that the confining lattice QCD potential lies roughly half-way between the pure Y-string and the pure Δ-string within the region of acute triangles. This allows us to set an upper and a lower bound on the value of the lattice coefficient: $-0.14 \leq v_{66} \leq -0.027$ using the values from Table I. Unfortunately, no $K = 3$ shell levels have been experimentally identified among the heavy-quark baryons as yet [12].

One possibility is to search for $K = 3$ states among light-quark baryons [12], which, however, involve significant contributions from relativity. In order to get a feeling for, or at least the sign of relativistic corrections, we shall employ a special case of the three-body problem: the extreme-relativistic three-body harmonic oscillator.

4. Extreme-relativistic three-body harmonic oscillator

The (extreme) relativistic three-quark Hamiltonian in configuration space is the $m_a \to 0$ limit of

$$H = \sum_a \sqrt{m_a^2 + \boldsymbol{p}_i^2} + \frac{k}{2}\left(\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2\right) \tag{7}$$

with the confining 3-body harmonic oscillator potential V_{HO}. The Hamiltonian in momentum space and CM frame reads

$$\tilde{H} = \frac{k}{2}\left(\frac{\partial^2}{\partial \boldsymbol{p}_{\boldsymbol{\rho}}^2} + \frac{\partial^2}{\partial \boldsymbol{p}_{\boldsymbol{\lambda}}^2}\right) + \sum_{i=1}^{3} |\boldsymbol{p}_i|\,,$$

which, after the substitutions $\boldsymbol{p}_{\boldsymbol{\rho}} \leftrightarrow \boldsymbol{\rho}$ and $\boldsymbol{p}_{\boldsymbol{\lambda}} \leftrightarrow \boldsymbol{\lambda}$, is equivalent to the (nonrelativistic) Schrödinger equation

$$\tilde{H}\tilde{\Psi} = \tilde{E}\tilde{\Psi}$$

for three identical particles with a mass $m = k$ and interacting with a linearly rising "CM-string" potential with unit string tension $\sigma = 1$. As we have developed hyperspherical harmonic methods [6, 7, 9] to deal with such three-body Schrödinger equations, we simply use the solution worked out in Ref. [11]. In Table I, we can see that the v_{66} coefficient has opposite signs in the Y- and Δ-string potentials, on the one hand, and the CM-string, *i.e.*, in the relativistic case and the Coulomb one, on the other. Thus we see that the relativistic effects, as well as the residual QCD Coulomb interaction, may reduce or even annihilate the Δ-string effects, which makes an identification of the confining potential more difficult.

5. Discussion and conclusions

We have shown how the lattice QCD data interpreted in terms of the hyperspherical variables can be used to put bounds on the mass-splittings of certain higher-lying odd-parity baryons using $O(6)$ hyperspherical harmonics.

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