EFFECTS OF THE MEAN FIELD ON FLUID DYNAMICS IN THE RELAXATION TIME APPROXIMATION*

Alina Czajka

National Centre for Nuclear Research, Pasteura 7, 02-093 Warsaw, Poland

Chun Shen

Department of Physics and Astronomy, Wayne State University
Detroit, MI 48201, USA
and
Department of Physics, Brookhaven National Laboratory
Upton, New York 11973-500, USA

Sigtryggur Hauksson, Sangyong Jeon, Charles Gale

Department of Physics, McGill University
3600 rue University, Montreal, Quebec H3A 2T8, Canada

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In this paper, the nonequilibrium correction to the distribution function containing a time- and space-dependent mass is obtained. Given that, fully consistent fluid dynamic equations are formulated. Then, the physics of the bulk viscosity is elaborated for the Boltzmann and Bose–Einstein gases within the relaxation-time approximation. It is found that the parametric form of the ratio $\zeta/\tau_{\rm R}$ for the quantum gas is affected by the infrared cut-off. This may be an indication that the relaxation-time approximation is too crude to obtain a reliable form of bulk viscosity.

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1. Introduction

Relativistic viscous hydrodynamics is a very efficient framework to investigate and understand the physics of strongly interacting matter created experimentally in heavy-ion collisions [1, 2]. Apart from the conservation

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laws and constraints on local thermal equilibrium, a viscous hydrodynamical description requires transport coefficients determined by the microscopic structure of a given system. Given that, different phenomena control parametric forms of different coefficients. In weakly interacting systems, the shear viscosity is mostly determined by kinetic energy scale, while bulk viscosity appears as a consequence of the conformal anomaly [3–5]. Due to the complexity of the symmetry breaking and importance of different energy scales, the bulk sector is still much less understood than the shear transport phenomena. In particular, it is important for modelling of heavy-ion collisions to have a fluid dynamics formulation where temperature-dependent mass is properly included in the bulk sector. This is very challenging in general but doable to some extent in the regime of the coupling constant, where analytic methods can be employed.

In this paper, we consider a dilute gas of weakly interacting particles of single species with the Bose–Einstein or Boltzmann statistics where effective kinetic theory is applicable and the mean-field effects can be systematically examined. Within the kinetic theory, many attempts were undertaken so far to provide such a description, see Refs. [6–13] but it seems they were incomplete. Hence we revisited the problem. The entire comprehensive examination of the consequences of the temperature-dependent mass on dynamics of the system is presented in our paper [14]. Here we only provide a very concise summary of the main results.

2. Nonequilibrium deviation from the equilibrium distribution function

The quasiparticle dynamics of a system of a single species is governed by the Boltzmann equation. When the x-dependence of the quasiparticle energy is known, the equation can be written as follows:

$$\left(\tilde{k}^\mu \partial_\mu - \tfrac{1}{2}\nabla \tilde{m}_x^2 \cdot \nabla_k\right) f = C[f]\,, \tag{1}$$

where $C[f]$ is the collision term and $f = f(x,k)$ is a distribution function of quasiparticles. $\tilde{k}^\mu = (\tilde{k}^0, \boldsymbol{k})$ is the quasiparticle four-momentum, where $\tilde{k}_0 \equiv \mathcal{E}_k$ is the nonequilibrium energy $\mathcal{E}_k = \sqrt{\boldsymbol{k}^2 + \tilde{m}_x^2}$. A time and space dependence appears in the mass definition $\tilde{m}_x^2 \equiv \tilde{m}^2(x) = m_0^2 + m_{\rm th}^2(x)$, where m_0 is the constant mass and $m_{\rm th}(x)$ is the nonequilibrium thermal mass, which varies in time and space. Note that we use tilde and calligraphic letters to denote nonequilibrium quantities. For a system in equilibrium, we ommit tilde and use standard letters so that the four-momentum, energy and mass of quasiparticles are denoted by k^μ, E_k and m_x, respectively. The thermal mass of quasiparticles in equilibrium is denoted by $m_{\rm eq}$ and the equilibrium phase-space density by f_0.

The phase-space density function $f(x,k)$ is the main object of the kinetic theory which carries information on the behavior of quasiparticles. When the departure from the equilibrium state is weak, the equilibration process is controlled by the small deviation in the distribution function

$$\Delta f(x,k) = f(x,k) - f_0(x,k) = \delta f(x,k) + \delta f_{\rm th}(x,k)\,, \tag{2}$$

where $f_0(x,k)$ is the equilibrium Bose–Einstein distribution function $f_0(x,k) = 1/(\exp\big(E_k(x)\beta(x)\big) - 1)$, where $\beta(x) = 1/T(x)$ with T being the temperature of the system. As can be seen, Δf has two parts: δf occurs because of hydrodynamic forces, which, in turn, changes the functional form of f_0, and $\delta f_{\rm th} = f_{\rm th} - f_0$, which is the effect of a small thermal mass deviation $\Delta m^2_{\rm th} = m^2_{\rm th} - m^2_{\rm eq}$. The function $f_{\rm th}$ has the local-equilibrium form of the Bose–Einstein distribution function

$$f_{\rm th}(x,k) \equiv f_0(x,k)|_{m_0^2+m^2_{\rm eq}(x)\to m_0^2+m^2_{\rm eq}(x)+\Delta m^2_{\rm th}(x)}$$

and by expanding it, one finds the correction $\delta f_{\rm th}$ expressed through $\Delta m^2_{\rm th}$. Since $\Delta m^2_{\rm th}$ is the nonequilibrium small deviation, which itself is a functional of Δf, the problem must be solved self-consistently. As a result, one gets

$$\Delta f = \delta f - T^2 \frac{{\rm d}m^2_{\rm eq}}{{\rm d}T^2} \frac{f_0(1+f_0)}{E_k} \frac{\int {\rm d}K \delta f}{\int {\rm d}K E_k f_0(1+f_0)}\,, \tag{3}$$

where ${\rm d}K = {\rm d}^3\boldsymbol{k}/[(2\pi)^3 E_k]$. In previous analyses [6–13], the second term in Eq. (3) was missing or was incomplete. The thermal mass of the quantum gas is given by $m^2_{\rm eq} = \lambda T^2/24$, where λ is the coupling constant assumed to be small. The temperature dependence of the thermal mass is found to be $T^2 \frac{{\rm d}m^2_{\rm eq}}{{\rm d}T^2} = m^2_{\rm eq} + T^2\beta_\lambda/48$, where $\beta_\lambda \equiv T\frac{{\rm d}\lambda}{{\rm d}T}$ is the renormalization group β_λ-function which controls the running of the coupling constant as a function of the energy scale. β_λ should be calculated via diagrammatic methods and in the case of the scalar theory, it is positive and proportional to λ^2.

3. Equations of hydrodynamics with thermal corrections

The stress-energy tensor of nonequilibrium fluid dynamics takes the following form:

$$T^{\mu\nu} = \int {\rm d}\mathcal{K} \tilde{k}^\mu \tilde{k}^\nu f - g^{\mu\nu} U\,, \tag{4}$$

where $g^{\mu\nu} = {\rm diag}(1,-1,-1,-1)$, U is the mean-field contribution and ${\rm d}\mathcal{K} \equiv {\rm d}^3\boldsymbol{k}/[(2\pi)^3\mathcal{E}_k]$ is the Lorentz-invariant measure. First, let us point out that

when the system is in equilibrium, the stress-energy tensor has the same form as Eq. (4) but all quantities are replaced by their equilibrium counterparts so that $\tilde{k}^\mu \to k^\mu$, $\mathcal{E}_k \to E_k$, $f \to f_0$, and $U \to U_0$.

It is essential to underline that the fluid dynamics equations with thermal effects discussed here are valid as long as all assumptions about quasiparticles of kinetic theory hold. Then, the departure of all quantities from its equilibrium forms is determined by small corrections. In particular, one has $f = f_0 + \Delta f$, where Δf is given by Eq. (3), and $U = U_0 + \Delta U$. The equilibrium mean-field contribution should satisfy $\mathrm{d}U_0 = \frac{1}{2}\mathrm{d}m_{\mathrm{eq}}^2 \int \mathrm{d}K f_0$, and the nonequilibrium correction is $\Delta U = \frac{1}{2}\Delta m_{\mathrm{th}}^2 \int \mathrm{d}K f_0$ to guarantee the energy-momentum conservation law $\partial_\mu T^{\mu\nu} = 0$. Consequently, the stress-energy tensor (4) may be decomposed into the local equilibrium part $T_0^{\mu\nu}$ and the nonequilibrium correction $\Delta T^{\mu\nu}$ as follows:

$$T^{\mu\nu} = T_0^{\mu\nu} + \Delta T^{\mu\nu}\,. \tag{5}$$

The equilibrium energy-momentum tensor has the familiar form of $T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}$, where u^μ is the four-velocity and $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. ϵ_0 is the energy density and P_0 is the local thermodynamic pressure, which are defined as

$$\epsilon_0 = \int \mathrm{d}K\, E_k^2 f_0 - U_0\,, \qquad P_0 = \frac{1}{3}\int \mathrm{d}K\, \boldsymbol{k}^2 f_0 + U_0\,. \tag{6}$$

The presence of the mean-field contribution in Eq. (6) does not change the enthalpy, $h_0 = \epsilon_0 + P_0$. One can also check that the thermodynamic relation $Ts_0 = T\mathrm{d}P_0/\mathrm{d}T = \epsilon_0 + P_0$, where s_0 is the entropy density, is fulfilled.

$\Delta T^{\mu\nu}$, which depends on Δf and ΔU, carries entire dynamical information needed to determine how the nonequilibrium system evolves into its equilibrium state. The Landau matching is defined by the eigenvalue problem, which in the fluid rest frame can be expressed by the conditions on the energy and the momentum densities $T^{00} = \epsilon$ and $T^{0i} = 0$, respectively. Given that, one defines the local equilibrium as the state having the same local energy and the momentum density, which is the essence of the Landau matching conditions found as

$$\Delta T^{00} = \int \mathrm{d}K \left[E_k^2 - T^2 \frac{\mathrm{d}m_{\mathrm{eq}}^2}{\mathrm{d}T^2}\right] \delta f = 0\,, \quad \Delta T^{0i} = \int \mathrm{d}K E_k k^i \Delta f = 0\,. \tag{7}$$

ΔT^{ij} can be manipulated and reorganized in such a way to separate the spin 0 part and the spin 2 part, $\Delta T^{ij} = \pi^{ij} + \delta^{ij}\Pi$, where the shear-stress tensor π^{ij} and the bulk pressure Π have commonly known forms

$$\pi^{ij} = \int \mathrm{d}K \left(k^i k^j - \frac{1}{3}\delta^{ij}\boldsymbol{k}^2\right)\delta f\,, \qquad \Pi = \frac{1}{3}\int \mathrm{d}K \boldsymbol{k}^2 \delta f\,. \tag{8}$$

4. Transport coefficients in the Anderson–Witting model

In the Anderson–Witting model, the Boltzmann equation with the x-dependent thermal mass is given by

$$\left(k^\mu \partial_\mu - \frac{1}{2}\partial_i m_{\rm eq}^2 \frac{\partial}{\partial k_i}\right) f_0(x,k) = -\frac{E_k}{\tau_{\rm R}} \Delta f(x,k)\,, \tag{9}$$

where $k^\mu = (E_k, \boldsymbol{k})$ and $\tau_{\rm R}$ is the relaxation time which is assumed to be energy-independent. Δf is the nonequilibrium correction given by Eq. (3) and we let $\delta f = f_0(1+f_0)\phi$, where $\phi = \phi_{\rm s} + \phi_{\rm b}$, that is, it consists of the shear and bulk part. Solving the Anderson–Witting model, one finds their forms to be

$$\phi_{\rm s}(k) \;=\; -\frac{\tau_{\rm R}}{TE_k}\left(k^i k^j - \frac{1}{3}\delta^{ij}\boldsymbol{k}^2\right)\partial_j u_i\,, \tag{10}$$

$$\phi_{\rm b}(k) \;=\; \tau_{\rm R}\beta\left(\partial_i u^i\right)\left(c_{\rm s}^2 - \frac{1}{3}\right)\left(E_k - \frac{1}{E_k}\frac{J_{3,0} - T^2\left({\rm d}m_{\rm eq}^2/{\rm d}T^2\right)J_{1,0}}{J_{1,0} - T^2({\rm d}m_{\rm eq}^2/{\rm d}T^2)J_{-1,0}}\right)\,, \tag{11}$$

where $c_{\rm s}^2$ is the speed of sound and the factor $c_{\rm s}^2 - 1/3$ depending both on the mass m_0^2 and β_λ fixes the nonconformality parameter. The thermodynamic functions $J_{n,q}$ are defined as follows: $J_{n,q} = 1/(2q+1)!! \int {\rm d}K (u\cdot k)^{n-2q} (-\Delta_{\mu\nu}k^\mu k^\nu)^q f_0(k)(1+f_0(k))$. One can check that with these forms of solution the energy of the system is conserved and the Landau matching conditions are satisfied. Having given solutions (10) and (11), one can use Eq. (8) to find shear-stress tensor and bulk pressure. Next, by comparing them with $\pi^{ij} = 2\eta\sigma^{ij}$, where $\sigma^{ij} = -1/2(\partial^i u^j + \partial^j u^i - 2/3 g^{ij}\partial_k u^k)$ and $\Pi = -\zeta\partial_i u^i$, the ratios $\eta/\tau_{\rm R}$ and $\zeta/\tau_{\rm R}$ can be extracted. Therefore, from the shear part, one finds the known form of the ratio $\eta/\tau_{\rm R} = (\epsilon_0 + P_0)/5$ and from the bulk part, one gets

$$\frac{\zeta}{\tau_{\rm R}} \;\approx\; T^4\left(\frac{1}{3} - c_{\rm s}^2\right)^2\left(\frac{2\pi^3 T}{25 m_x} - \frac{4\pi^2}{75}\left(1 - \frac{9m_{\rm eq}^2}{8m_x^2}\right)\right)\,, \tag{12}$$

where $m_x = \sqrt{m_0^2 + m_{\rm eq}^2(x)}$. The ratio for the Boltzmann statistics $f_{0,c}(k) = {\rm e}^{-\beta E_k}$ can be found analogously, and it is

$$\frac{\zeta_{\rm Boltz}}{\tau_{\rm R}} \approx T^4\left(\frac{1}{3} - c_{\rm s}^2\right)^2\left(\frac{60}{\pi^2} - \frac{36 m_x}{\pi T}\right)\,. \tag{13}$$

Note that the structure of expression (12) is slightly different from the one in (13) due to the factor T/m_x. The origin of this difference comes from the fact that the infrared limit of the Bose–Einstein factor behaves like $f_0(k) \sim T/E_k$, while the Boltzmann factor does not show such a behavior.

5. Conclusions

In this paper, we examined the effects of mean field on fluid dynamics. We found the correction to the distribution function which enabled us to formulate fully consistent equations of fluid dynamics as well as to solve the Anderson–Witting model to compute $\zeta/\tau_{\rm R}$ of the Bose–Einstein and Boltzmann gases. The ratio $\zeta/\tau_{\rm R}$ for the Boltzmann gas has a parametrically expected form, that is, it is given by the nonconformality parameter squared. In the case of the Bose–Einstein gas, the leading order term of $\zeta/\tau_{\rm R}$ has an additional energy scale-dependent factor T/m_x. We suspect that it is an indication that the relaxation-time approximation applied here is too crude to get the expected form of the ratio since the constant relaxation time is insensitive to the soft scale.

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REFERENCES

[1] C. Gale, S. Jeon, B. Schenke, *Int. J. Mod. Phys. A* **28**, 1340011 (2013).
[2] U. Heinz, R. Snellings, *Annu. Rev. Nucl. Part. Sci.* **63**, 123 (2013).
[3] S. Jeon, *Phys. Rev. D* **52**, 3591 (1995).
[4] S. Jeon, L.G. Yaffe, *Phys. Rev. D* **53**, 5799 (1996).
[5] P.B. Arnold, C. Dogan, G.D. Moore, *Phys. Rev. D* **74**, 085021 (2006).
[6] C. Sasaki, K. Redlich, *Phys. Rev. C* **79**, 055207 (2009).
[7] P. Chakraborty, J.I. Kapusta, *Phys. Rev. C* **83**, 014906 (2011).
[8] M. Bluhm, B. Kämpfer, K. Redlich, *Phys. Rev. C* **84**, 025201 (2011).
[9] P. Romatschke, *Phys. Rev. D* **85**, 065012 (2012).
[10] M. Albright, J.I. Kapusta, *Phys. Rev. C* **93**, 014903 (2016).
[11] P. Chakraborty, J.I. Kapusta, *Phys. Rev. C* **95**, 014907 (2017).
[12] L. Tinti, A. Jaiswal, R. Ryblewski, *Phys. Rev. D* **95**, 054007 (2017).
[13] M. Alqahtani, M. Nopoush, R. Ryblewski, M. Strickland, *Phys. Rev. Lett.* **119**, 042301 (2017).
[14] A. Czajka *et al.*, *Phys. Rev. C* **97**, 044914 (2018).