GAUGE-COVARIANT DIAGONALIZATION OF πa_1 MIXING AND THE RESOLUTION OF A LOW-ENERGY THEOREM* **

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Using a recently proposed gauge covariant diagonalization of πa_1-mixing, we show that the low-energy theorem $F^\pi = ef_\pi^2 F^{3\pi}$ of current algebra, relating the anomalous form factor $F_{\gamma\to\pi^+\pi^0\pi^-} = F^{3\pi}$ and the anomalous neutral pion form factor $F_{\pi^0\to\gamma\gamma} = F^\pi$, is fulfilled in the framework of the Nambu–Jona-Lasinio (NJL) model, solving a long-standing problem encountered in the extension including vector and axial-vector mesons. At the heart of the solution is the presence of a $\gamma\pi\bar{q}q$ vertex which is absent in the conventional treatment of diagonalization and leads to a deviation from the vector meson dominance (VMD) picture. It contributes to a gauge-invariant anomalous tri-axial (AAA) vertex as a pure surface term.

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The Wess–Zumino [1] effective action, with topological content clarified by Witten [2], describes all effects of QCD anomalies in low-energy processes with photons and Goldstone bosons, without reference to massive vector mesons. The extension to the case with spin-1 mesons is not unique, and has been addressed in different frameworks [3–5]. Important issues arise when one includes the spin-1 states. Here, we address the concept of VMD and the pseudoscalar–axial-vector mixing (πa_1 mixing) of meson states. In particular, it has been shown in [4] that the *complete* VMD is not valid in either $\pi^0 \to \gamma\gamma$ or $\gamma \to 3\pi$ processes, and that mixing affects hadronic amplitudes in [6, 7]. Therefore, one should demonstrate how the departure from VMD occurs and how πa_1 mixing is treated in order to comply with the predictions of the Wess–Zumino action. This is not a trivial task, in [8], it has been reported that in a number of well-known models [9–16], the πa_1 mixing breaks low-energy theorems (LET) for some anomalous processes, *e.g.*, $\gamma \to 3\pi$, $K^+K^- \to 3\pi$. In [17], based on the gauge covariant treatment of πa_1 mixing, only recently addressed [18–22], we show precisely how the deviation of the complete VMD occurs in the framework of the NJL Lagrangian, fulfilling the LET

$$F^\pi = e f_\pi^2 F^{3\pi} \,. \tag{1}$$

The procedure is sufficiently general to be applied in other processes.

To be more definite, recall that the πa_1 diagonalization is generally performed by a linearized transformation of the axial vector field

$$a_\mu \to a_\mu + \frac{\partial_\mu \pi}{a g_\rho f_\pi} \,, \tag{2}$$

where $\pi = \tau_i \pi^i$, $a_\mu = \tau_i a_\mu^i$ and τ_i are the SU(2) Pauli matrices; $g_\rho \simeq \sqrt{12\pi}$ is the coupling of the ρ meson to two pions, and $f_\pi \simeq 93$ MeV is the pion weak decay constant. In extensions of the model that couple to the electroweak sector, this replacement violates gauge invariance [18–22] in anomalous processes, leaving, however, the real part of the action invariant [20, 21]. For example, the anomalous low-energy amplitude describing the $a_1 \to \gamma\pi^+\pi^-$ decay is not transverse [18, 19]. To restore gauge invariance, the gauge covariant derivative $\mathcal{D}_\mu \pi$ must be used instead of $\partial_\mu \pi$ [18–22]

$$a_\mu \to a_\mu + \frac{\mathcal{D}_\mu \pi}{a g_\rho f_\pi} \,, \qquad \mathcal{D}_\mu \pi = \partial_\mu \pi - ie A_\mu [Q, \pi] \,, \qquad Q = \frac{1}{2}\left(\tau_3 + \frac{1}{3}\right) . \tag{3}$$

In the context of the LET, $F^\pi = e f_\pi^2 F^{3\pi}$ mixing occurs related to both anomalous form factors, but it has been proven in [17] that the radiative decay $\pi^0 \to \gamma\gamma$ is not affected by the mixing, and coincides with the low-energy result of current algebra given by the Lagrangian density [1, 2]

$$\mathcal{L}_{\pi\gamma\gamma} = -\frac{1}{8} F^{\pi} \pi^0 e^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\,, \qquad F^{\pi} = \frac{N_c e^2}{12\pi^2 f_\pi}\,, \tag{4}$$

where e is the electric charge, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ stands for the strength of the electromagnetic field, N_c is the number of quark colors. The absence of mixing is seen as follows. In the NJL model, one can switch to spin-1 variables without direct photon–quark coupling, as described in the VMD picture. Then $\mathcal{L}_{\pi\gamma\gamma}$ is related to the $\pi^0\omega\rho$ quark triangle shown in Fig. 1 (a). At leading order of a derivative expansion, the current-algebra result $\Gamma(\pi^0 \to \gamma\gamma) = 7.1$ eV is obtained. Diagram 1 (b), due to mixing, is described by an axial-vector vector vector (AVV) Adler–Bell–Jackiw anomaly [23–26]. The related surface term (ST) which results from the difference of two linearly divergent amplitudes is *a priori* arbitrary. Here, this arbitrary parameter is fixed on gauge-invariant grounds of $a_1 \to \gamma\gamma$, upon which graph 1 (b) vanishes at leading order of a derivative expansion. This complies with the Landau–Yang theorem [27, 28] which states that a massive unit spin particle cannot decay into two on-shell massless photons.

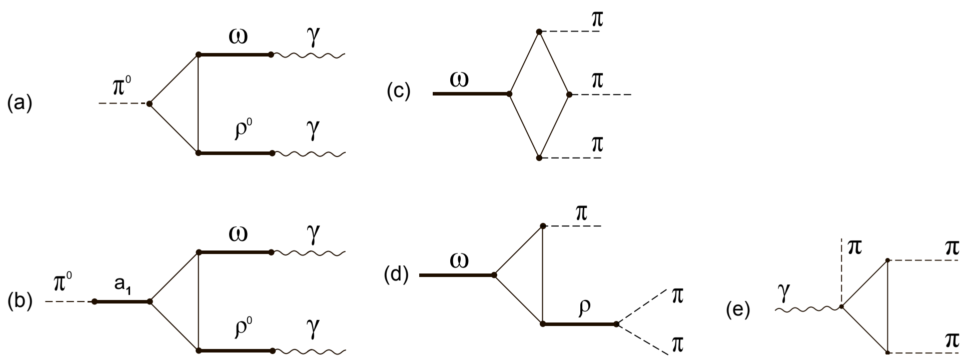

Fig. 1. (a) and (b): the two graphs describing the $\pi^0 \to \gamma\gamma$ decay in the NJL model, (b) for πa_1-mixing effects on the pion line. Quark loop contributions to $\omega \to 3\pi$ decay, (c) full set of possible diagrams without and with 1, 2, and 3 πa_1-mixing effects on the pion line (not drawn); (d) ρ exchange diagrams without and with πa_1 transitions. (e) contribution to $\gamma \to 3\pi$ decay due to covariant πa_1 diagonalization, see (3), with pion lines subject to πa_1 mixing.

Effects of πa_1 mixing in $\gamma \to 3\pi$ amplitudes (due to G-parity, it is sufficient to consider the isoscalar component of the photon related to $\omega \to 3\pi$) have been studied in detail by Wakamatsu [8], using prescription (2). He found that the amplitude of the $\omega \to 3\pi$ decay contains uncompensated contributions generated by πa_1 mixing, breaking the LET at the order of $1/a^2$, where $a = \frac{m_\rho^2}{g_\rho^2 f_\pi^2} = 1.84$ and m_ρ is the empirical mass of the ρ-meson. This conclusion is based on the assumption that VMD is valid.

Let us recall and complement the calculations made in [8]. The diagrams contributing to the $\omega \to 3\pi$ decay are shown in Fig. 1 (c), (d), where we have additionally taken into account the box diagram with three πa_1 transitions in (c) as well as the contribution of the $\omega\rho(a_1 \to \pi)$ vertex in the ρ-exchange graph (d), both neglected in [8]. The corresponding amplitude is given by

$$A_{\omega\to 3\pi} = -\frac{N_c g_\rho}{4\pi^2 f_\pi^3} e_{\mu\nu\alpha\beta}\epsilon^\mu(q) p_0^\nu p_+^\alpha p_-^\beta F_{\omega\to 3\pi}\,, \tag{5}$$

where p_0, p_+, p_- are the momenta of the pions, $\epsilon^\mu(q)$ the polarization of the ω-meson with momentum q, and the form factor $F_{\omega\to 3\pi}$ is found to be

$$F_{\omega\to 3\pi} = \left(1 - \frac{3}{a} + \frac{3}{2a^2} + \frac{1}{8a^3}\right) + \left(1 - \frac{c}{2a}\right) \sum_{k=0,+,-} \frac{g_\rho^2 f_\pi^2}{m_\rho^2 - (q-p_k)^2}\,. \tag{6}$$

In the first parentheses, the box diagrams without, with one, two, and three πa_1 transitions are given correspondingly. The last term represents the contribution of ρ-exchange graphs, where c controls the magnitude of an arbitrary local part of the anomalous AVV-quark-triangle. In the low-energy limit, the sum yields $3/a$, as one neglects the dependence on momenta in (6), leading to full cancellation among the terms of the order of $1/a$, as is well known [8]. The ST c contributes at the order of $1/a^2$. For $c = 0$, we reproduce the πa_1-mixing effect found in [8] to this order. Had c been used instead to cancel the πa_1-mixing effect, as $c = 1 + 1/(12a)$, a too low width $\Gamma(\omega \to \pi^+\pi^0\pi^-) = 3.2$ MeV would have been obtained as compared to experiment $\Gamma(\omega \to \pi^+\pi^0\pi^-) = 7.57 \pm 0.13$ MeV. Furthermore, the value $c = 0$ is also required following [29], where the chiral Ward identities (WI) for $\gamma \to 3\pi$ imply that both the chiral triangle and the box anomaly contribute as

$$A^{\text{tot}}_{\gamma\to 3\pi} = \frac{3}{2} A^{\text{AVV}} - \frac{1}{2} A^{\text{VAAA}}\,, \tag{7}$$

where $A^{\text{tot}}_{\gamma\to 3\pi}$, A^{AVV} and A^{VAAA} are, respectively, the total $\gamma\pi\pi\pi$ amplitude, the $\gamma \to \omega \to \pi\rho \to \pi\pi\pi$ process and the point $\gamma \to \omega \to \pi\pi\pi$ amplitude. This result is consistent with both the chiral WI and with the KSFR relation [30, 31], which arises in the NJL model at $a = 2$. One sees from Eq. (6) that, if one neglects the terms of the order of $1/a^2$ and higher in the box contribution and puts $c = 0$ in the ρ-exchange term, the amplitude A^{VAAA} has a factor $(1 - 3/a) = -1/2$, and the A^{AVV} amplitude has a factor $(1 - c/(2a))3/a = 3/2$, as is required by the chiral WI. On the other hand, if c is chosen to cancel πa_1-mixing effects, these amplitudes contribute with relative weights $-7/64$ and $71/64$, respectively. Therefore, the ST c cannot be used to resolve the πa_1-mixing puzzle, the chiral WI require $c = 0$. This pattern has been considered in [3, 5, 8], and reproduces well the phenomenological value of the width. That allows us to conclude, following [8], that if the VMD is a valid theoretical hypothesis, the $\gamma \to \omega \to 3\pi$ amplitude contains contributions due to πa_1 mixing that violate the LET (1)

$$A_{\gamma\to 3\pi} = -F^{3\pi} e_{\mu\nu\alpha\beta}\epsilon^{\mu}(q)p_0^{\nu}p_+^{\alpha}p_-^{\beta}, \tag{8}$$

$$F^{3\pi} = \frac{N_c e}{12\pi^2 f_\pi^3}\left(1+\frac{3}{2a^2}+\frac{1}{8a^3}\right) \neq \frac{N_c e}{12\pi^2 f_\pi^3}. \tag{9}$$

In the following, we will show that it is possible to combine the phenomenologically successful value $c = 0$ with a full cancellation of πa_1-mixing effects within the NJL approach by taking into account the anomalous AAA triangle shown in Fig. 1 (e), which occurs as result of (3)

$$\begin{aligned} A &= \frac{N_c e}{4a^3 f_\pi^3}\left\{p_-^{\sigma}\left[J_{\mu\nu\sigma}(p_0,p_-) - J_{\mu\sigma\nu}(p_-,p_0)\right]\right. \\ &\quad \left. + p_+^{\sigma}\left[J_{\mu\nu\sigma}(p_0,p_+) - J_{\mu\sigma\nu}(p_+,p_0)\right]\right\}\epsilon^{\mu}(q)p_0^{\nu}. \end{aligned} \tag{10}$$

The low-energy expansion of the loop integral $J_{\mu\nu\sigma}$ starts from a linear term

$$J_{\mu\nu\sigma}(p_0,p_-) = \frac{1}{24\pi^2} e_{\mu\nu\sigma\rho}\left(p_0 - p_- - 3\upsilon\right)^{\rho} + \ldots \tag{11}$$

Owing to the shift ambiguity related to the formal linear divergence of this integral, the result depends on the undetermined 4-vector υ_ρ

$$A = -\frac{N_c e}{4\pi^2 f_\pi^3} e_{\mu\nu\sigma\rho}\epsilon^{\mu}(q)p_0^{\nu}(p_+ + p_-)^{\sigma}\left(\frac{\upsilon^{\rho}}{4a^3}\right). \tag{12}$$

This is the complete result for this triangle diagram. The 4-vector υ_ρ is represented as a linear combination of the independent momenta of the process, $\upsilon_\mu = b_1 q_\mu + b_2(p_+ - p_-)_\mu + b_3(p_+ + p_-)_\mu$, but only the second term survives in (12). Thus, the graph shown in Fig. 1 (d) gives an additional contribution $\Delta F^{3\pi}$ to the form factor $F^{3\pi}$

$$\Delta F^{3\pi} = \frac{N_c e}{12\pi^2 f_\pi^3}\left(\frac{-3b_2}{2a^3}\right), \tag{13}$$

where b_2 is dimensionless and as yet undetermined. This constitutes a further example in which an arbitrary-regularization-dependent parameter should be fixed by the physical requirements [26, 32, 33]. The AAA amplitude would have been zero had it been regularized in advance by any regularization that sets ST to zero. For a detailed discussion of this and further anomalous vertices appearing in the present calculation, we refer to [17]. To fix b_2, we use the LET (1); requiring that the unwanted terms in (9) vanish, we find that $b_2 = a + \frac{1}{12} = 1.92$. Thus, the solution of the πa_1-mixing problem in the $\gamma \to 3\pi$ amplitude can be associated with the ST of the anomalous non-VMD diagram shown on the right of Fig. 1.

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