# POLARIZATION-VORTICITY COUPLING WITHIN THE FLUID DYNAMICS WITH SPIN*

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We review the kinetic-theory-based derivation of relativistic hydrodynamics for polarized systems of particles with spin 1/2. In the case of global equilibrium, we find that the equivalence between the polarization and vorticity is not a necessary condition. With the use of the pseudogauge transformation, we show how to relate the de Groot–van Leeuwen–van Weert expressions forming the basis of our approach with the canonical currents resulting from the Noether theorem.

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## 1. Introduction

A first positive results of $\Lambda$ hyperon spin polarization measurements were reported recently by the STAR Collaboration [1–4]. The latter findings have intensified theoretical studies on the possible relation between the local vorticity of the matter created in relativistic heavy-ion collisions and the spin polarization of hadrons emitted in these reactions [5–35]; see also [36, 37] for recent reviews. So far, the thermal-model-based approaches with spin [7, 9] which correctly address the global polarization observables fail at explaining the differential measurements [4]. The main feature of these models, namely the assumption of the direct coupling between the polarization and thermal vorticity [6], increase their predictive power. However, at the same time, these approaches eliminate possibility of an independent dynamical evolution of spin during the system's expansion. Such a scenario was proposed first and studied in Refs. [38–41], and leads to the formulation of the framework of relativistic hydrodynamics with spin. In this work, we comment on the missing polarization-vorticity coupling in this approach, and the relation between this approach and the canonical formulation. Throughout the text, we use natural units with $c = k_{\mathrm{B}} = \hbar = 1$.

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## 2. Spin-polarized systems of particles in equilibrium

Let us consider a relativistic system of massive spin-1/2 particles (+) and antiparticles (−) in the local equilibrium state described by the following phase-space density matrices in spin space $(r,s=1,2)$ [6]:

$$f^{+}_{rs}(x,p)=\bar{u}_r(p)X^{+}u_s(p)\,,\qquad f^{-}_{rs}(x,p)=-\bar{v}_s(p)X^{-}v_r(p)\,,\tag{1}$$

with $X^{\pm}=\exp\left[\pm\xi(x)-\beta_\mu(x)p^\mu\pm\frac{1}{2}\omega_{\mu\nu}(x)\Sigma^{\mu\nu}\right]$, where $x$ and $p$ denote the space-time position and four-momentum, respectively. In Eqs. (1), $u_r(p)$ and $v_r(p)$ are the Dirac bispinors with the normalization $\bar{u}_r(p)u_s(p)=\delta_{rs}$ and $\bar{v}_r(p)v_s(p)=-\delta_{rs}$. The quantities $\beta^\mu\equiv U^\mu/T$ and $\xi\equiv\mu/T$, with $T$, $\mu$ and $U^\mu$ denoting the temperature, baryon chemical potential and four-velocity, respectively, are the usual Lagrange multipliers introduced to guarantee energy, linear momentum and baryon number conservation. In order to conserve total angular momentum in the system, we also introduce the so-called spin polarization tensor $\omega^{\mu\nu}=-\omega^{\nu\mu}$ which couples to the Dirac spin operator $\Sigma^{\mu\nu}\equiv\frac{i}{4}[\gamma^\mu,\gamma^\nu]$.

Using Eqs. (1) in the kinetic-theory definitions of Ref. [42], one can derive corresponding expressions for the equilibrium Wigner functions

$$\mathcal{W}^{\pm}_{\rm eq}(x,k)=\frac{{\rm e}^{\pm\xi}}{4m}\int {\rm d}P\,{\rm e}^{-\beta\cdot p}\;\delta^{(4)}(k\mp p)\times\left[2m(m\pm\not{p})\cosh(\zeta)\pm\frac{\sinh(\zeta)}{2\zeta}\,\omega_{\mu\nu}\,(\not{p}\pm m)\Sigma^{\mu\nu}(\not{p}\pm m)\right],\tag{2}$$

where $k$ is the off-mass-shell four-momentum of particles, ${\rm d}P={\rm d}^3p/((2\pi)^3E_p)$ is the on-mass-shell invariant momentum integration measure with $E_p=\sqrt{m^2+\boldsymbol{p}^2}$ and $\zeta=\frac{1}{2\sqrt{2}}\sqrt{\omega_{\mu\nu}\omega^{\mu\nu}}$.

Hereafter, we consider the following Clifford-algebra expansion of the equilibrium Wigner function given by Eq. (2):

$$\mathcal{W}^{\pm}_{\rm eq}(x,k)=\frac{1}{4}\left[\mathcal{F}^{\pm}_{\rm eq}(x,k)+i\gamma_5\mathcal{P}^{\pm}_{\rm eq}(x,k)+\gamma^\mu\mathcal{V}^{\pm}_{{\rm eq},\mu}(x,k)+\gamma_5\gamma^\mu\mathcal{A}^{\pm}_{{\rm eq},\mu}(x,k)+\Sigma^{\mu\nu}\mathcal{S}^{\pm}_{{\rm eq},\mu\nu}(x,k)\right].\tag{3}$$

The real coefficient functions $\mathcal{X}\in\{\mathcal{F},\mathcal{P},\mathcal{V}_\mu,\mathcal{A}_\mu,\mathcal{S}_{\nu\mu}\}$ in Eq. (3) can be obtained by tracing $\mathcal{W}^{\pm}_{\rm eq}(x,k)$ multiplied first by the Clifford-algebra generators $\Gamma\in\{\mathbf{1},-i\gamma_5,\gamma_\mu,\gamma_\mu\gamma_5,2\Sigma_{\mu\nu}\}$.

## 3. Kinetic equations

Let us consider the Wigner function which satisfies the kinetic equation

$$\left[\gamma_\mu\left(k^\mu+\frac{i\hbar}{2}\,\partial^\mu\right)-m\right]\mathcal{W}(x,k)=C[\mathcal{W}(x,k)]\,.\tag{4}$$

In global equilibrium, Eq. (4) is satisfied exactly with $C[\mathcal{W}(x,k)] = 0$. Using semi-classical expansion in powers of $\hbar$ of the coefficients $\mathcal{X}$ at next-to-leading order, one finds the following kinetic equations for $\mathcal{F}_{\rm eq}$ and $\mathcal{A}^\nu_{\rm eq}$:

$$k^\mu \partial_\mu \mathcal{F}_{\rm eq}(x,k) = 0\,, \qquad k^\mu \partial_\mu \mathcal{A}^\nu_{\rm eq}(x,k) = 0\,, \qquad k_\nu \mathcal{A}^\nu_{\rm eq}(x,k) = 0\,. \tag{5}$$

The coefficient functions $\mathcal{P}, \mathcal{V}_\mu$, and $\mathcal{S}_{\nu\mu}$ may be calculated from $\mathcal{F}_{\rm eq}$ and $\mathcal{A}^\nu_{\rm eq}$ using the algebraic relations found in Ref. [40]. In the case of global equilibrium, Eqs. (5) are fulfilled exactly, which requires $\beta_\mu$ to be a Killing vector, and $\xi$ and $\omega_{\mu\nu}$ to be constant. However, in this case, $\omega_{\mu\nu}$ is not necessarily equal to the so-called thermal vorticity $\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right) = \text{const}$.

## 4. Hydrodynamic equations

In local equilibrium, instead of demanding that Eqs. (5) are satisfied exactly, we follow a standard approach [43–47] and assume that certain moments in momentum space of Eqs. (5) are fulfilled. This method yields hydrodynamic equations which express conservation laws of baryon number, energy and linear momentum, and angular momentum [40]

$$\partial_\mu N^\mu = 0\,, \tag{6}$$

$$\partial_\mu T^{\mu\nu}_{\rm GLW} = 0\,, \tag{7}$$

$$\partial_\lambda S^{\lambda,\alpha\beta}_{\rm GLW} = 0\,, \tag{8}$$

respectively. The net baryon current $N^\mu$, the energy-momentum tensor $T^{\alpha\beta}_{\rm GLW}$, and the spin current $S^{\alpha,\beta\gamma}_{\rm GLW}$ are given by the de Groot–van Leeuwen–van Weert (GLW) [42] forms

$$\begin{aligned} N^\alpha &= nU^\alpha\,, \\ T^{\alpha\beta}_{\rm GLW} &= \varepsilon U^\alpha U^\beta - P\Delta^{\alpha\beta}\,, & (9)\\ S^{\alpha,\beta\gamma}_{\rm GLW} &= \cosh(\xi)\Big[n_{(0)}U^\alpha\omega^{\beta\gamma} + \mathcal{A}_{(0)}\,U^\alpha U^\delta U^{[\beta}\omega^{\gamma]}{}_\delta & (10)\\ &\quad + \mathcal{B}_{(0)}\Big(U^{[\beta}\Delta^{\alpha\delta}\omega^{\gamma]}{}_\delta + U^\alpha\Delta^{\delta[\beta}\omega^{\gamma]}{}_\delta + U^\delta\Delta^{\alpha[\beta}\omega^{\gamma]}{}_\delta\Big)\Big]\,, \end{aligned}$$

with $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$ being the projector on the space orthogonal to the flow four-vector.

In the situations when the polarization is small, which is the case observed in experiment, the baryon density, the energy density, and the pressure are given by the formulas: $n = 4\,\sinh(\xi)\,n_{(0)}(T)$, $\varepsilon = 4\,\cosh(\xi)\,\varepsilon_{(0)}(T)$, and $P = 4\,\cosh(\xi)\,P_{(0)}(T)$, respectively. The auxiliary functions $n_{(0)}$, $\varepsilon_{(0)}$, and $P_{(0)}$ are the corresponding quantities describing thermodynamic properties of the system of neutral massive Boltzmann particles without spin

[48, 49], while the quantities $\mathcal{B}_{(0)}$ and $\mathcal{A}_{(0)}$ are given by the expressions $\mathcal{B}_{(0)} = -\frac{2}{\hat{m}^2} s_{(0)}(T)$, and $\mathcal{A}_{(0)} = -3\mathcal{B}_{(0)} + 2n_{(0)}(T)$, with $s_{(0)} = \left(\varepsilon_{(0)} + P_{(0)}\right)/T$ denoting the entropy density and $\hat{m} = m/T$.

## 5. Relating the GLW and canonical tensors

It is instructive to check that the expressions for the GLW currents given in Eqs. (9) and (10) are directly related to the corresponding quantities resulting from the canonical formulation using Noether theorem. The relation between the two is given by the so-called pseudogauge transformation given by the relations

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can} \tag{11}$$

and

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \tfrac{1}{2}\partial_\lambda \left( \Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can} \right), \tag{12}$$

where the superpotential is given by $\Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$. One can check that the canonical energy-momentum is not symmetric, hence the spin current is not in this case conserved separately

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x)\,. \tag{13}$$

## 6. Summary

In this work, we briefly reviewed the kinetic-theory-based method to derive the framework of relativistic hydrodynamics with spin. For that purpose, we employed the definitions of the phase-space density matrices in spin space proposed in Ref. [6] and used the de Groot–van Leeuwen–van Weert expressions to derive corresponding equilibrium Wigner functions. Starting with the kinetic equation for the Wigner function, we used the semi-classical expansion to obtain the transport equations for the Clifford algebra scalar and axial-vector coefficient functions. Subsequently, employing the standard method of moments, we derived the hydrodynamic equations with spin. In the global equilibrium, we find that the polarization-thermal-vorticity coupling is not a necessary condition, as it is usually argued in thermal-model approaches. Finally, we find that our approach may be related to the canonical framework using the pseudogauge transformation.

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