

DECONFINEMENT AND HADRON RESONANCE GAS FOR HEAVY QUARKS*

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I discuss the deconfinement transition in $(2+1)$ -flavor QCD in terms of Polyakov loops as well as the hadron resonance gas for hadrons containing static quarks and charm quarks.

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1. Introduction

Heavy quarks and infinitely heavy (static) quarks play an important role when discussing deconfinement transition in strongly interacting matter at high temperatures. The early works on deconfinement considered the free energy of static quark Q as well as the free energy of static quark–antiquark ($Q\bar{Q}$) pair [1–3], and the lattice calculations of these quantities was a focus of many works, see Ref. [4] for a historic review. Deconfinement is closely related to color screening. The production rate of quarkonia, bound states of a heavy quark and antiquark, was suggested as a probe of deconfinement in heavy-ion collisions [5]. The basic idea behind this proposal was that the color screening in the deconfined medium effects the binding of heavy quarks (see also Ref. [6] for a review). Recent lattice QCD studies, however, mostly focus on the chiral aspects of the transition at high temperature, see *e.g.* Refs. [7, 8] for recent reviews. In this contribution, I will discuss the deconfinement in $(2+1)$ -flavor QCD with (almost) physical quark masses in terms of Polyakov loops in different renormalization schemes.

The Hadron Resonance Gas (HRG) model has been used to understand the thermodynamics below the cross-over temperature for many years [9–16]. The HRG model received significantly less attention for static and heavy quarks, for example for thermodynamics of charm quarks. I will discuss the application of HRG model for these cases, namely the renormalized Polyakov loop and entropy of the static quark, and for the charm-baryon number correlations.

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2. The renormalized Polyakov loop and deconfinement

The expectation value of the Polyakov loop defined as

$$\langle L \rangle = \left\langle \frac{1}{N} \text{Tr} \mathcal{P} \exp \left(ig \int_0^{1/T} dx_0 A_0(\mathbf{x}, x_0) \right) \right\rangle \quad (1)$$

is the order parameter for deconfinement in $SU(N)$ gauge theory. It is related to the free energy of a static quark $L = \exp(-F_Q/T)$ [1–3]. The correlator of the Polyakov loop

$$C_{\text{PL}}(r, T) = \langle L(\mathbf{x}) L^\dagger(0) \rangle \quad (2)$$

is related to the free energy of a static $Q\bar{Q}$ pair at a distance of $r = |\mathbf{x}|$ [3], $C_{\text{PL}}(r, T) = \exp(-F_{Q\bar{Q}}(r, T)/T)$. Color screening in the deconfined phase implies $C_{\text{PL}}(r \rightarrow \infty, T) = |\langle L \rangle|^2$ or $F_{Q\bar{Q}}(r \rightarrow \infty, T) = 2F_Q(T)$, with F_Q being finite. In the confined (hadronic) phase of $SU(N)$ gauge theory, F_Q is infinite and thus $\langle L \rangle = 0$. In QCD, $\langle L \rangle$ is no longer an order parameter since F_Q can be finite in the hadronic phase, where it is determined by the binding energy of the static-light and static-strange hadrons. Despite the fact that the Polyakov loop is not an order parameter in QCD, it is still useful for understanding the screening properties of the medium, as will be discussed below. In fact, the Polyakov loop and its correlator play a key role for the non-perturbative understanding of chromo-electric screening¹.

The free energy of static $Q\bar{Q}$ pair can be renormalized by requiring that at very short distances, it coincides with the zero temperature $Q\bar{Q}$ potential [19]. This way one also gets the renormalized value of F_Q . The renormalized Polyakov loop in (2+1)-flavor QCD is shown in Fig. 1 and compared to the corresponding results in $SU(3)$ [19, 20] and $SU(2)$ [21] gauge theories. The calculations in (2 + 1)-flavor QCD have been performed using HISQ action with physical value of the strange-quark mass and light-quark masses corresponding to the pion mass of 161 MeV in the continuum limit [22]. Most of the lattice calculations discussed in this contribution have been obtained using this setup. The continuum extrapolation has been performed [22] for the Polyakov loop. I also show the continuum (2 + 1)-flavor results obtained using stout action with physical quark masses [23]. We see that the behavior of the Polyakov loop in the vicinity of the transition temperature is quite different in QCD and $SU(N)$ gauge theories: L_{ren} behaves smoothly in QCD and is quite small. An interesting feature of the renormalized Polyakov loop

¹ An alternative approach to study chromo-electric screening in terms of gauge fixed chromo-electric gluon propagators has also been proposed [17, 18].

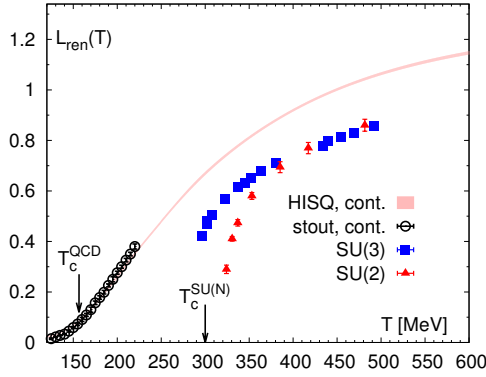


Fig. 1. The renormalized Polyakov loop in $(2+1)$ -flavor QCD in the continuum limit compared to the Polyakov loop in $SU(2)$ and $SU(3)$ gauge theories. The arrows indicate the approximate positions of the QCD chiral crossover and the phase transition temperature in $SU(2)$ and $SU(3)$ gauge theory.

is the fact that it is larger than one at high temperatures (see Fig. 1) and approaches one from above. This feature can be easily understood if we recall that high temperature also corresponds to very short distances, where the zero temperature $Q\bar{Q}$ potential is given by leading order perturbative result: $-C_F\alpha_s/r$, and thus is negative. This means that $F_{Q\bar{Q}}(r, T)$ is also negative at short distances. Due to color screening, $F_{Q\bar{Q}}(r, T)$ cannot increase indefinitely with increasing r , implying that at high enough temperature, it will be negative for all distances, *cf.* Fig. 1 (left) in Ref. [24]. This also means that $F_Q < 0$, *i.e.* $L_{\text{ren}} > 1$.

A gradient flow provides an alternative way to renormalize the Polyakov loop [22, 25, 26]. The gradient flow is an evolution of the original gauge fields defined on the 4D lattices in a fictitious time, called the flow time, according to 5D classical equation of motion [27]. This evolution smears the gauge fields in a radius $f = \sqrt{8t}$ and thus removes the UV component of the fields. It can be thought as continuous smearing of the gauge fields. In order to avoid distortion of thermal physics and to obtain renormalized results, the flow time should satisfy the condition $a \ll f \ll 1/T$ [25]. In Fig. 2, I show the renormalized Polyakov loop obtained using the gradient flow for flow time $f = f_0 = 0.2129$ fm, and for different temporal extent $N_\tau = 1/(aT)$. The fact that there is no significant N_τ dependence in the figure means that the cutoff dependence is removed. In the crossover region, the flow-time dependence of the Polyakov loop is very mild for $f \geq f_0$. Let us also notice that the Polyakov loop defined using the gradient flow is smaller than one. This is expected as the renormalized Polyakov loop in this case is given by the trace of an $SU(3)$ matrix constructed from the smeared links. The tem-

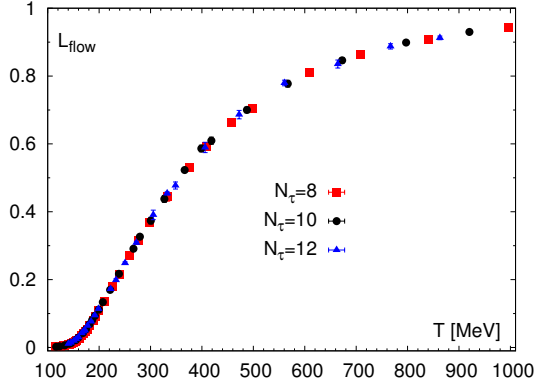


Fig. 2. The Polyakov loop calculated with gradient flow for flow time $f = f_0$ at different N_τ .

perature dependence of the free energy obtained in this scheme is the same as in the conventional renormalization discussed above, the difference amounts to a temperature-independent additive constant [22, 25, 26, 28]. The gradient flow also reduces the statistical noise, which allows to calculate the Polyakov loop not only for static quarks but also for static charges in higher representations of SU(3) group (sextet, octet *etc.*). The Polyakov loops in higher representations are not order parameters for deconfinement even for SU(N) gauge theories. Nonetheless, they have similar temperature dependence as the Polyakov loop in the fundamental representation in QCD and are sensitive to color screening, and thus to deconfinement. In the crossover, the flow time dependence of the Polyakov loop in higher representations is somewhat larger, and stable results can be only obtained for $f \geq 2f_0$ [26]. The Polyakov loops in higher representations satisfy the so-called Casimir scaling for $T > 300$ MeV [25], meaning the corresponding free energies are proportional to the Casimir operators of the respective representation. This is expected in perturbation theory. In fact, Casimir scaling violations first could show up at the order of α_s^4 [29].

Since the gradient flow reduces the statistical errors and at the same time renormalizes composite operators, it can be used to study the renormalized Polyakov loop susceptibilities. Following Ref. [30], three types of susceptibilities can be defined

$$\chi = (VT^3) (\langle L^2 \rangle - \langle L \rangle^2), \quad (3)$$

$$\chi_L = (VT^3) (\langle (\text{Re}L)^2 \rangle - \langle L \rangle^2), \quad \chi_T = VT^3 \langle (\text{Im}L)^2 \rangle. \quad (4)$$

Note that in QCD $\langle \text{Im}L \rangle = 0$. The flow dependence of χ and χ_L turns out to be significant [22]. For $f = 3f_0$, these quantities show a peak around $T \simeq 180\text{--}200$ MeV. In Fig. 3, I show the lattice results for χ_T . This quantity also

has a significant flow time dependence. For $f = 3f_0$, it has a peak around the chiral cross-over temperature. The flow time dependence is largely reduced in the ratio χ_T/χ [22]. This ratio has a characteristic decrease around the chiral cross-over temperature [22]. In this sense, χ_T may be sensitive to deconfinement in QCD.

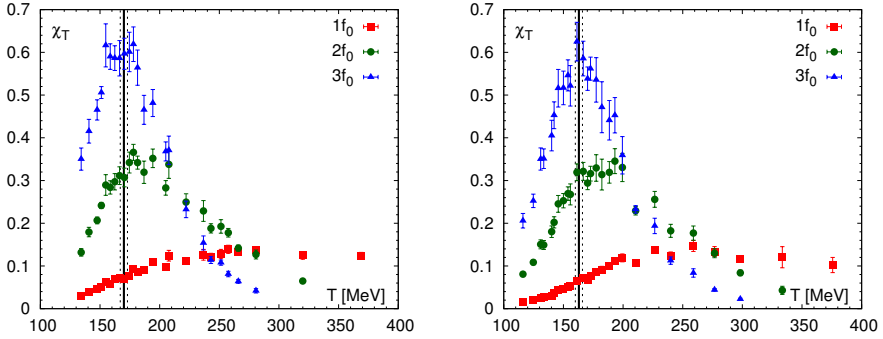


Fig. 3. The Polyakov loop susceptibility χ_T calculated for flow times $f = f_0, 2f_0$ and $3f_0$ on $N_\tau = 6$ lattice (left) and on $N_\tau = 8$ lattice (right).

At low temperatures, one may try to understand the Polyakov loop in terms of a gas of static-light and static-strange hadrons [31, 32]. Hadrons consisting of a static quark will interact with other hadrons in the medium and one can assume that these interactions can be taken care of by including static-light and static-strange resonances in the spirit of the HRG model [32]. Lattice QCD calculations provide information on the few low-lying static light hadron states, which is not sufficient. We could use charm and beauty hadrons as proxies for static-light and static-strange hadrons once the finite quark effect has been taken into account [32]. However, the knowledge of the spectrum of charm and beauty hadrons is rather incomplete. Therefore, in order to include higher excited states, one needs to use the quark model results [33–36] for the hadron spectrum [32]. To compare F_Q obtained in such an HRG model with the continuum extrapolated lattice results, one needs to shift it by an additive constant [32]. The comparison of the lattice and the HRG result shows that HRG can only describe the free energy of a static quark for $T < 140$ MeV. At higher temperatures, F_Q shows a qualitatively different behavior. In particular, it has an inflection point around the chiral cross-over temperature [32]. Since F_Q is a physical quantity, this inflection point could be related to the deconfinement transition. Unlike the inflection point for L_{ren} , it does not depend on the choice of the renormalization scheme. The entropy of a static quark is defined as

$$S_Q = -\frac{\partial F_Q}{\partial T}. \quad (5)$$

The inflection point in F_Q corresponds to a maximum in S_Q . The comparison of the lattice results and the HRG results is simpler for S_Q because the additive normalization constant drops out. The comparison of the lattice results with HRG for S_Q is shown in Fig. 4. The temperature axis in the figure has been rescaled by $T_c = 156.5$ MeV [37] for the HRG result and by 159.5 MeV for the $(2+1)$ -flavor QCD results, since the lattice calculations have been performed for $m_l = m_s/20$ instead of the physical value $m_l = m_s/27$. This results in 3 MeV upward shift in T_c according to the analysis of Ref. [38]. The lattice result for S_Q clearly disagrees with HRG. The entropy of the static quark shows a peak around the chiral cross-over temperature. For comparison, we also show lattice results for S_Q from SU(3) gauge theory [19, 20], 3-flavor QCD [39], and 2-flavor QCD [40] with larger than the physical quark mass. The temperature variables in the corresponding lattice results have been rescaled by the phase transition temperature of SU(3) gauge theory, and by the chiral cross-over temperature for 3-flavor and 2-flavor QCD for the respective quark masses. For SU(3) gauge theory, the divergence in S_Q for $T \rightarrow T_c^+$ is clearly related to the deconfinement transition. For 3-flavor and 2-flavor QCD at larger than physical quark masses, we know that the chiral crossover temperature and the deconfinement temperature defined in terms of Polyakov loop susceptibility coincide [41]. Thus, here it is also justified to associate the peak in S_Q with the deconfinement transition. From these considerations, we conclude that the peak in S_Q in $(2+1)$ -flavor QCD is also associated with the deconfinement temperature, and the chiral and deconfinement transitions coincide in that sense for the

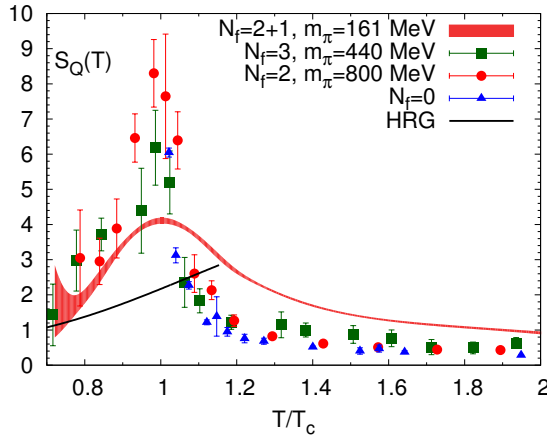


Fig. 4. (Color online) The continuum estimate for the entropy of a static quark, S_Q in $(2+1)$ -flavor QCD (gray/red band) compared to the corresponding results in 3-flavor and 2-flavor QCD with unphysical quark masses as well as in SU(3) gauge theory (zero flavor QCD), see the text.

physical value of the quark masses. Another interesting question is whether the critical behavior in $(2+1)$ -flavor QCD for $m_l \rightarrow 0$ has an imprint on the Polyakov loop expectation value. Very recent lattice calculations suggest that this is indeed the case [42]. Since the chiral phase transition temperature $T_c = 132(+6)(-3)$ MeV [43] is significantly smaller than the chiral cross-over temperature for physical quark masses, it is possible that F_Q as a function of temperature has two inflection points².

3. Charm-baryon number correlations

Charm-quark fluctuations and charm-baryon number correlations up to the fourth order have been studied in $(2+1)$ -flavor QCD on the lattice using the HISQ action [44]. The charm fluctuations and charm-baryon number correlations are defined through the derivatives of the QCD pressure with respect to the corresponding chemical potentials as

$$\chi_n^C = \frac{\partial^n (p/T^4)}{\partial (\mu_C/T)^n}, \quad \chi_{nm}^{BC} = \frac{\partial^{n+m} (p/T^4)}{\partial (\mu_C/T)^n \partial (\mu_B/T)^m}. \quad (6)$$

Since charm quarks are heavy, only $|C| = 1$ sector contributes to the above quantities in the temperature range of interest. The ratio $\chi_{13}^{BC}/\chi_{22}^{BC}$ is sensitive to deconfinement. Below the crossover, this ratio is one because in the HRG framework the relevant degrees of freedom are singly charmed baryons, $|B| = |C| = 1$. At high temperatures, this ratio approaches three, since the relevant degrees of freedom are charm quarks. The lattice calculations confirm these expectations [44]. Let me mention that the above ratio is convenient because the lattice artifacts as well as the uncertainties related to the tuning of the charm-quark mass cancel out. For a detailed comparison with HRG, it is also convenient to consider a ratio for the same reasons. The relevant ratio in this case is $\chi_{13}^{BC}/(\chi_2^C - \chi_{13}^{BC})$, which can be considered as a proxy for charm-baryon to charm-meson pressure [44]. As mentioned before, the charmed-hadron spectrum is not very well known. This is especially the case for charmed baryons. Therefore, if one only includes charm hadrons from Particle Data Group, the HRG largely under-predicts the lattice results for $\chi_{13}^{BC}/(\chi_2^C - \chi_{13}^{BC})$. If one includes additional charmed hadrons from quark models, as discussed for the static quark free energy, the lattice results agree with HRG below and in the vicinity of the chiral crossover [44].

Another interesting question is the nature of charm degrees of freedom above the chiral cross-over temperature but for $T < 250$ MeV. In this temperature region, the baryon number charm correlations are not described by HRG but are also significantly smaller than in the quark gas [44]. One may

² I thank F. Karsch for raising this point during my talk.

wonder if charm-hadron-like excitations can explain this feature. It is known that charmonia can exist above the deconfinement phase transition temperature [45, 46]. There are some indications that hadron-like excitations may exist in the deconfined phase also in the light-quark sector [47–49]. Therefore, in Ref. [50], it was proposed that the lattice results on baryon number charm correlations can be understood if charm-hadron-like excitations exist above T_c . According to this model, charm quark are dominant degrees of freedom only for $T > 200$ MeV [50].

4. Conclusions

From the discussions above, it is clear that the Polyakov loop behaves quite differently in QCD and $SU(N)$ gauge theory. In particular, the renormalized Polyakov loop in QCD is small at the crossover temperature irrespective of the renormalization scheme and shows a smooth behavior. The gradient flow can be used to renormalize the Polyakov loop and study its fluctuations, as well as Polyakov loops in higher representations. The Polyakov loop susceptibilities do not show the type of behavior seen in $SU(N)$ gauge theories and, therefore, cannot be used to define the deconfinement transition temperature. On the other hand, the entropy of a static quark can be used to identify the deconfinement temperature through its maximum. The entropy of the static quark has a peak around the chiral cross-over temperature. In this sense, the chiral and deconfinement transitions coincide for the physical values of the quark masses. The HRG model fails to describe the free energy of static quarks for $T > 140$ MeV despite the fact that many resonances have been included in the analysis. This may be due to the deconfinement physics encoded in the Polyakov loop. The charm-baryon number correlations can be described by the HRG model once additional hadron states, that are not yet observed experimentally but predicted by the quark model, are included. In this respect, charm and static quarks are quite different. For $T_c < T < 250$ MeV, the baryon-charm correlations cannot be described by the HRG model but are very different from the quark gas expectations. This feature may be explained if one assumes charm-hadron-like excitations to exist above the cross-over temperature.

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