MODELING THE DENSITY OF STATES*

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We examine the influence of various S-matrix structures: poles, roots, and Riemann sheets, on the density of states.

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1. Introduction

The S-matrix formulation of statistical mechanics [1, 2] offers a unique approach to study how the interactions among constituents determine the bulk properties of the medium. The essential connection is provided by the density of states (DoS) [3, 4], which can be expressed in terms of the S-matrix via

$$\begin{aligned} B(E) &= \frac{1}{2}\mathrm{Im\,Tr}\left[S^{-1}\frac{\mathrm{d}}{\mathrm{d}E}S - \left(\frac{\mathrm{d}}{\mathrm{d}E}S^{-1}\right)S\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}E}\,\mathrm{Im}\,\ln\det S(E)\,. \end{aligned} \tag{1}$$

The thermal partition function is given by an integral of the DoS with the appropriate Boltzmann weight. For example, the thermal pressure due to the interaction can be computed as

$$\Delta P = \int \frac{\mathrm{d}E}{2\pi}\,B(E)\,P^{(0)}(E,T)\,,$$

where $P^{(0)}(E = m_a, T)$ denotes the pressure of an ideal gas of particles with mass m_a. Other thermal (and fluctuation) observables can be similarly computed [5, 6].

The advantage in writing the DoS in terms of the S-matrix is clear: The S-matrix has a direct connection to (existing and future) experimental data, accompanied by many powerful theoretical tools, *e.g.* chiral perturbation

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theory [7, 8], lattice QCD [9], and effective models [10–12]. It is also a good theoretical framework to discuss interactions. In these proceedings, we shall examine, within a simple coupled-channel model, how the various structures of particle dynamics would influence the DoS.

2. DoS of an interacting system

2.1. Single channel

If the interaction is dominated by a single resonance (non-relativistic) of mass $m_{\rm res}$ and width γ, the resonant phase shift can be written as

$$\delta_{\rm res}(E) = \tan^{-1} \frac{\gamma(E)/2}{m_{\rm res} - E} \,. \tag{2}$$

The effective spectral function B assumes the standard Breit–Wigner form upon neglecting the energy dependence of the numerator $\gamma(E) \to \gamma_{\rm bw}$

$$\begin{aligned} B_{\rm res}(E) &= 2 \frac{\rm d}{{\rm d}E}\, \delta_{\rm res}(E) \\ &\approx \frac{\gamma_{\rm bw}}{(E - m_{\rm res})^2 + \gamma_{\rm bw}^2/4} \\ &= -2\, {\rm Im}\, \frac{1}{E - m_{\rm res} + i\, \gamma_{\rm bw}/2} \\ &= \frac{\rm d}{{\rm d}E}\, {\rm Im}\, \ln \left(\frac{m_{\rm res} + i\, \gamma_{\rm bw}/2 - E}{m_{\rm res} - i\, \gamma_{\rm bw}/2 - E} \right) . \end{aligned} \tag{3}$$

The last two lines make clear the connection to the resolvent [3]. When the empirical phase shift from scattering experiments is used, both resonant and non-resonant interactions are correctly incorporated, and the result becomes insensitive to the choice of parameters in an individual model.

2.2. Coupled-channel system

As energy increases, new interaction channels open up and the scattering becomes inelastic. The prescription of Eq. (1) remains valid, but the S-matrix should now be formally identified as a matrix acting in the open-channel space, *i.e.* an $N_{\rm chan} \times N_{\rm chan}$ matrix [3, 4, 6]. From this, an effective phase shift $\mathcal{Q}$ can be constructed

$$\mathcal{Q}(E) = \frac{1}{2} \int\limits_{E_{\rm ref}}^{E} {\rm d}E'\, B\left(E'\right) = \frac{1}{2}\, {\rm Im}\, \ln \det \left(\frac{S(E)}{S(E_{\rm ref})} \right) . \tag{4}$$

This is an essential simplification: a single effective phase shift function to describe the whole multi-channel system.

3. Improving the HRG

The hadron resonance gas (HRG) model [13] provides a simple scheme for incorporating resonances in $B(E)$ (see Eq. (3))

$$\det S(E) = \prod_{\{\text{res}\}} \frac{z^{\star}_{\text{res}} - E}{z_{\text{res}} - E}\,, \tag{5}$$

where {res} is a table of resonances (*e.g.* from PDG) approximated as simple poles

$$z_{\text{res}} \approx m_{\text{res}} - i\,0^{+}\,. \tag{6}$$

$\mathcal{Q}_{\text{HRG}}$ is then given by a sum of step functions

$$\mathcal{Q}(E) \to \mathcal{Q}_{\text{HRG}}(E) = \sum_{\text{res}} d_{IJ} \times (\pi \times \theta(E - m_{\text{res}}))\,. \tag{7}$$

The finite widths of resonances can be easily accounted for, *e.g.*, via

$$z_{\text{res}} \to m_{\text{res}} - i\,\gamma(E)/2\,. \tag{8}$$

To study $\det S$ in an interacting system, we compute it over the complex plane for a simple coupled-channel model describing the $(\pi\pi, K\bar{K})$ system. An important feature of the model is that resonances are dynamically generated instead of being explicitly introduced in the Hamiltonian. See Ref. [4] for details. The result is shown in Fig. 1, with the phase shift shown in Fig. 2. Here, we highlight some key features:

1. Five resonance poles are identified in the model [4], distributed across the Riemann sheets II (3 poles) and III (2 poles, not shown). The DoS is only strongly influenced by some of them: *e.g.* $p1$ and $p2$, but not $p3$. This is understood by considering the continuity of the phase of $\det S$ across the Riemann sheets: $p3$ is far away from the physical line as sheet I and sheet II are no longer connected above the $K\bar{K}$ threshold. In the latter case, the relevant poles are those in sheet III.

2. Roots are important in determining the DoS. They encode details of non-resonant interactions. We find that substantial contribution comes from root $r1$. Unfortunately, many studies report only the locations of poles but not for roots.

3. It is obvious that the distribution of poles and roots in the interacting system does not follow scheme (5), *i.e.* a sum of step functions will not adequately describe the phase shift function $\mathcal{Q}$. This is true even after the widths of resonances are included.

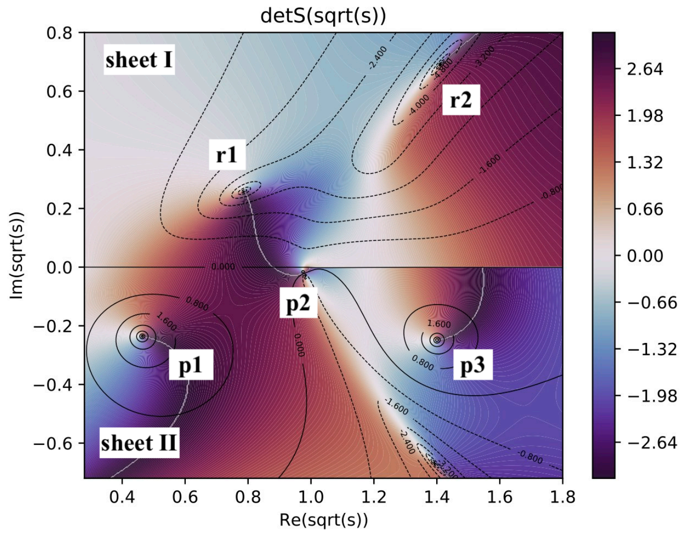

Fig. 1. (Color online) Landscape of S-matrix phase $\ln\det S(\sqrt{s}\,)$ of the $\pi\pi, K\bar{K}$ coupled-channel system on the energy sheets I and II. Color signifies the value of the phase angle and contour lines specify magnitudes of $\ln\,|\det S|$. Poles (roots) are characterized by the clockwise (anti-clockwise) rotation of the color phase and by a large, positive (negative) values of $\ln\,|\det S|$ reflected in the contour lines. The physical line is identified with the real line in sheet I $(\mathrm{Re}\,(\sqrt{s}\,) + i\,0^+)$.

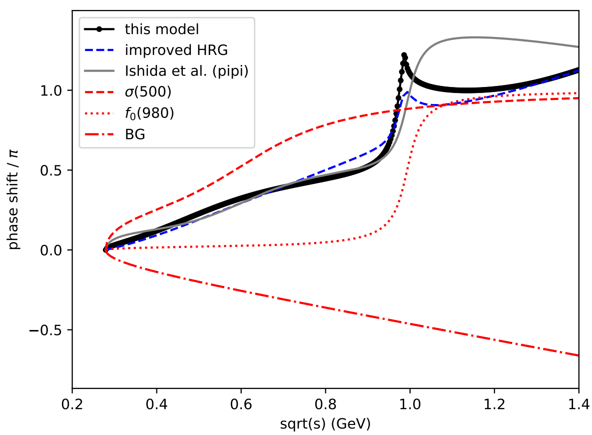

Fig. 2. The effective phase shift extracted from the coupled-channel model [4], compared to the result of the model in Ref. [14]. For the latter, the decomposition of its components: $\sigma(500)$, $f_0(980)$ and a repulsive background, is also displayed. The improved HRG scheme is from Eq. (9).

To describe the phase shift, one can construct an HRG-like scheme, but selecting only the relevant poles and roots. For energy below the $K\bar{K}$ threshold, we pick

$$\det S(E) \approx \frac{(r1 - E)(r2 - E)}{(p1 - E)(p2 - E)}\,. \tag{9}$$

Compared to the bare scheme (5), the (subtractive) correction would appear as a repulsive force. See Fig. 2.

4. In the work of Ref. [14], a repulsive background is introduced, in addition to the resonant shifts of $\sigma(500)$ and $f_0(980)$, to reproduce the experimental phase shift. In the current model, such a subtractive correction comes from the mismatch between $r1$ and $p1$, giving a weaker phase motion. In both models, we see the suppression of $\sigma(500)$ within the $I = 0$ channel. A repulsive component is also expected from the t- and u-channel exchanges in chiral perturbation theory.

   Despite the different decomposition into resonances and backgrounds, both models roughly capture the experimental phase shift. This illustrates a powerful feature of the method: when the relevant experimental results are available to quantify the DoS, the thermal observables computed become model independent.

4. Going further

The S-matrix framework is flexible in that the degrees of freedom (DoFs) used in the Hamiltonian can be different from those appearing in the S-matrix (open channels). In particular, resonances and other dynamically generated states, are taken into account only via the latter. Such a separation would become interesting when quarks and gluons DoFs are employed in the Hamiltonian. Presumably quarks and gluons are forbidden in the open channels, and at low temperatures, probing low energies due to the Boltzmann suppression, should yield a gas of pions. Realizing this in the S-matrix approach could yield novel insights into describing the thermal properties of interacting hadrons and eventually the deconfinement phase transition in QCD.

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