VARIATIONAL FORMULATIONS OF GENERAL RELATIVITY*

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In this paper, I present a few results devoted to variational methods in General Relativity Theory. I will focus on the equivalence between three different variational formulations. Moreover, I analyze how the dependence on covariant derivatives affects the affine connection.

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1. Introduction

To properly describe the General Relativity Theory, it is necessary to use the variational calculus. It is quite surprising that there exist a few possibilities (so-called *pictures*) to do this, so let me present three of them:

— metric picture:

$$\mathcal{L}_g = \mathcal{L}_g \left(g_{\mu\nu},\, g_{\mu\nu,\alpha}\,, g_{\mu\nu,\alpha\beta},\, \phi,\, \phi_{,\nu} \right) \,,$$

— Palatini picture:

$$\mathcal{L}_{\mathrm{P}} = \mathcal{L}_{\mathrm{P}} \left(g_{\mu\nu},\, \Gamma^{\kappa}{}_{\lambda\mu},\, \Gamma^{\kappa}{}_{\lambda\mu,\nu},\, \phi,\, \phi_{,\nu} \right) \,,$$

— affine picture:

$$\mathcal{L}_{\mathrm{A}} = \mathcal{L}_{\mathrm{A}} \left(\Gamma^{\kappa}{}_{\lambda\mu},\, \Gamma^{\kappa}{}_{\lambda\mu,\nu},\, \phi,\, \phi_{,\nu} \right) \,,$$

where Γ is a symmetric but <u>not necessarily</u>(!) metric connection and ϕ represents the general (tensor) matter field. The main topic of this paper is to convince the reader that all of them are equivalent *on shell* — on the subspace which is defined by the Euler–Lagrange system. Moreover, if the Lagrangian depends on the connection via the covariant derivatives of matter field, it leads to the non-trivial extension of Einstein equations and symmetric (non-metric) affine connection.

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2. Standard metric picture

In this picture, the fundamental object is the metric Lagrangian $\mathcal{L}_g$ which is a sum of two parts: Hilbert Lagrangian $\mathcal{L}_{\rm H}$ and matter Lagrangian $\mathcal{L}_{\rm matt}$. It is necessary to mention that the Hilbert Lagrangian [9] presented below

$$\mathcal{L}_{\rm H} = \frac{\sqrt{|\det g|}}{16\pi}\,\overset{\circ}{R}$$

is constructed by the connection which is metric(!) *a priori*. Furthermore, all geometrical objects which are made by the metric connection $\overset{\circ}{\Gamma}$ will have the "∘" sign above. The metricity of the connection looks like an extra assumption and, in the most of well-known theories (*e.g.* electrodynamics), is absolutely correct.

The variation of the Hilbert Lagrangian was calculated in [6]

$$\delta\mathcal{L}_{\rm H} = -\frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\delta g_{\mu\nu} + \partial_\nu\left(\pi_\kappa{}^{\lambda\mu\nu}\,\delta\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\right) \tag{1}$$

$$= -\frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\delta g_{\mu\nu} + \pi^{\mu\nu}\,\delta\overset{\circ}{R}_{\mu\nu} + \left(\overset{\circ}{\nabla}_\nu\pi_\kappa{}^{\lambda\mu\nu}\right)\delta\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\,, \tag{2}$$

where $\overset{\circ}{\mathcal{G}}{}^{\mu\nu}$ is the Einstein tensor density constructed by the metric connection $\overset{\circ}{\Gamma}$ and π is the following function of the metric tensor g:

$$\pi_\kappa{}^{\lambda\mu\nu} := \delta^\nu_\kappa\,\pi^{\lambda\mu} - \delta^{(\lambda}_\kappa\,\pi^{\mu)\nu}\,, \tag{3}$$

$$\pi^{\mu\nu} := \frac{\sqrt{|\det g|}}{16\pi}\,g^{\mu\nu}\,. \tag{4}$$

Due to these definitions, the quantity $\overset{\circ}{\nabla}\pi$ is obviously zero, but we will use equation (2) in the next part of this paper.

The typical matter Lagrangian has the following variational structure (see [3, 8]):

$$\mathcal{L}_{\rm matt} = \mathcal{L}_{\rm matt}\left(\phi,\,\phi_{,\nu},\,g_{\mu\nu}\right)\,, \tag{5}$$

$$\delta\mathcal{L}_{\rm matt} = \frac{\partial\mathcal{L}_{\rm matt}}{\partial g_{\mu\nu}}\,\delta g_{\mu\nu} + \partial_\nu\left(p^\nu\,\delta\phi\right)\,. \tag{6}$$

It means that this Lagrangian density represents a special class of theories which do not depend upon covariant derivatives of matter field (*e.g.* theory of scalar field, electrodynamics, continuous media).

Now, the variation $\delta\mathcal{L}_g = \delta(\mathcal{L}_{\rm H}+\mathcal{L}_{\rm matt})$ is a sum of variations (2) and (6)

$$\delta\mathcal{L}_g = \left(\frac{\partial\mathcal{L}_{\rm matt}}{\partial g_{\mu\nu}} - \frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\right)\delta g_{\mu\nu} + \partial_\nu\left(p^\nu\,\delta\phi + \pi_\kappa{}^{\lambda\mu\nu}\,\delta\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\right)\,. \tag{7}$$

This formula gives us the Einstein equation as a consequence of vanishing the volume (bulk) term [3, 8]. In addition, it defines the symmetric energy-momentum tensor density $\mathcal{T}^{\mu\nu}$ as follows:

$$\frac{1}{2}\,\mathcal{T}^{\mu\nu} := \frac{\partial \mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}} = \frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\,. \tag{8}$$

The last term in boundary part $\partial\left(\pi\,\delta\,\overset{\circ}{\Gamma}\right)$ could be transformed like in Hilbert variation (1) implying the final variational formula for metric Lagrangian *on shell*

$$\begin{aligned}\delta\mathcal{L}_g &= \partial_\nu\left(p^\nu\,\delta\phi\right) + \pi^{\mu\nu}\,\delta\,\overset{\circ}{R}_{\mu\nu} + \left(\overset{\circ}{\nabla}_\nu \pi_\kappa{}^{\lambda\mu\nu}\right)\,\delta\,\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu} \\ &= \partial_\nu\left(p^\nu\,\delta\phi\right) + \delta\mathcal{L}_{\mathrm{H}} + \frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\,\delta g_{\mu\nu}\,.\end{aligned}$$

Subtracting $\delta\mathcal{L}_{\mathrm{H}}$ produces variation of the matter Lagrangian

$$\delta\mathcal{L}_{\mathrm{matt}} = \partial_\nu\left(p^\nu\,\delta\phi\right) + \frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu}\,\delta g_{\mu\nu} + \left(\overset{\circ}{\nabla}_\nu \pi_\kappa{}^{\lambda\mu\nu}\right)\,\delta\,\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}\,. \tag{9}$$

This formula strictly leads us to the Euler–Lagrange system (field equations)

$$p^\lambda = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\phi_{,\lambda}}\,, \tag{10}$$

$$\partial_\lambda p^\lambda = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\phi}\,, \tag{11}$$

$$\frac{1}{16\pi}\,\overset{\circ}{\mathcal{G}}{}^{\mu\nu} = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}}\,, \tag{12}$$

$$\overset{\circ}{\nabla}_\nu \pi_\kappa{}^{\lambda\mu\nu} = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\,\overset{\circ}{\Gamma}{}^\kappa{}_{\lambda\mu}} = 0\,, \tag{13}$$

where p^λ is momenta canonically conjugated to the matter field ϕ.

Unfortunately, this construction works only for the Lagrangians which do not depend on covariant derivatives. If the theory allows the dependence on the covariant derivatives of the matter fields, then the system (10)–(13) has to be reformulated. Moreover, it is quite hard and time-consuming in the metric picture. The remedy for such a problem is the Palatini variational formalism, which is described in the next section.

3. Palatini picture

The Palatini Lagrangian is made of the two parts: $\mathcal{L}_{\mathrm{H}}$ and $\mathcal{L}_{\mathrm{matt}}$ as before. In the metric picture, the connection $\overset{\circ}{\Gamma}$ was only a combination of the first derivatives and metric tensor itself. However, in the Palatini picture, the connection Γ and the metric tensor g are <u>independent</u> fields in variational sense. Then, equation (2) still holds but the metric connection $\overset{\circ}{\Gamma}$ has to be replaced by the general symmetric connection Γ. Additionally, the matter Lagrangian depends on the covariant derivatives of matter field. It means that the variation of this Lagrangian has the form written below:

$$\begin{aligned} \mathcal{L}_{\mathrm{matt}} &= \mathcal{L}_{\mathrm{matt}}\left(\phi,\, \nabla_\nu\phi,\, g_{\mu\nu}\right) , \\ \delta\mathcal{L}_{\mathrm{matt}} &= \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}}\,\delta g_{\mu\nu} + \mathcal{P}_\kappa{}^{\lambda\mu}\,\delta\Gamma^\kappa{}_{\lambda\mu} + \partial_\nu\left(p^\nu\,\delta\phi\right) , \end{aligned} \tag{14}$$

and

$$\mathcal{P}_\kappa{}^{\lambda\mu} := \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\Gamma^\kappa{}_{\lambda\mu}}\,, \qquad p^\nu := \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\nabla_\nu\phi} = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial\phi_{,\nu}}\,.$$

Hence, the variation of Palatini Lagrangian is the following:

$$\begin{aligned} \delta\mathcal{L}_{\mathrm{P}} &= \left(\frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}} - \frac{1}{16\pi}\mathcal{G}^{\mu\nu}\right)\delta g_{\mu\nu} + \left(\mathcal{P}_\kappa{}^{\lambda\mu} - \nabla_\nu\pi_\kappa{}^{\lambda\mu\nu}\right)\delta\Gamma^\kappa{}_{\lambda\mu} \\ &\quad + \partial_\nu\left(p^\nu\,\delta\phi + \pi_\kappa{}^{\lambda\mu\nu}\,\delta\Gamma^\kappa{}_{\lambda\mu}\right) . \end{aligned} \tag{15}$$

From this formula, we could extract two the most important (for our case) equations

$$\frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}} = \frac{1}{16\pi}\mathcal{G}^{\mu\nu}\,, \tag{16}$$

$$\mathcal{P}_\kappa{}^{\lambda\mu} = \nabla_\nu\pi_\kappa{}^{\lambda\mu\nu}\,. \tag{17}$$

The first one is the Einstein equation which describes the interaction between geometry of spacetime and matter — it is analogous to equation (12) in the metric picture. The second one shows that if our matter Lagrangian (in Palatini picture!) depends on covariant derivatives, the connection <u>must be</u> non-metric.

4. Affine picture

The affine picture is using Lagrangians where configurations are the first jets of the general symmetric connection Γ and the first jets of the matter field ϕ. Then,

$$\mathcal{L}_{\mathrm{A}} = \mathcal{L}_{\mathrm{A}} = \mathcal{L}_{\mathrm{A}}\left(\Gamma^{\kappa}{}_{\lambda\mu},\, \Gamma^{\kappa}{}_{\lambda\mu,\nu},\, \phi,\, \phi_{,\nu}\right) , \tag{18}$$

$$\delta\mathcal{L}_{A} = \partial_{\nu}\left(\pi_{\kappa}{}^{\lambda\mu\nu}\,\delta\Gamma^{\kappa}{}_{\lambda\mu} + p^{\nu}\,\delta\phi\right) . \tag{19}$$

Comparing this variational formula with (15) we can see that they are absolutely equivalent *on shell* — on the subspace spanned by the field equations (see Introduction). The only problem is in the different initial configurations between these two pictures. The Palatini Lagrangian explicitly depends on the metric opposite to the affine Lagrangian where metric is, via tensor density π (3), (4), a momentum canonically conjugated with the connection Γ. Fortunately, the Einstein equation (16) allows to rewrite the metric tensor g as a (usually implicit) function of other variables. Hence, implementation of such a metric into the $\mathcal{L}_{\mathrm{P}}$ produces the proper affine Lagrangian for this theory [3, 5].

As it was written at the beginning of this section, $\mathcal{L}_{\mathrm{A}}$ depends on the first jet of the connection Γ. Due to the fact that Γ and $\partial\Gamma$ are not tensors, they cannot occur in the Lagrangian randomly. Especially, the derivatives of the connection are organized (in this paper) in symmetric Ricci tensor $K_{\mu\nu}$ — for general connection, the skew-symmetric part of Ricci may not vanish. This assumption provides the following variational formula [3, 5]:

$$\delta\mathcal{L}_{\mathrm{A}} = \left(\nabla_{\nu}\pi_{\kappa}{}^{\lambda\mu\nu}\right)\,\delta\Gamma^{\kappa}{}_{\lambda\mu} + \pi^{\mu\nu}\delta K_{\mu\nu} + \partial_{\nu}\left(p^{\nu}\,\delta\phi\right) .$$

Now, equations (16), (17) from the Palatini picture may be equivalently represented as

$$\pi^{\mu\nu} = \frac{\partial\mathcal{L}_{\mathrm{A}}}{\partial K_{\mu\nu}} , \qquad \nabla_{\nu}\pi_{\kappa}{}^{\lambda\mu\nu} = \frac{\partial\mathcal{L}_{\mathrm{A}}}{\partial\Gamma^{\kappa}{}_{\lambda\mu}} . \tag{20}$$

The important difference between the affine picture and the remaining ones is the fact that the affine Lagrangian cannot be split into Hilbert and matter parts.

The simple example of this picture is the Lagrangian for the vacuum gravity with cosmological constant Λ [2, 3]

$$\mathcal{L}_{\mathrm{A}} = \frac{\sqrt{|\det K|}}{8\pi\Lambda} ,$$

with field equations (20):

$$K_{\mu\nu} = \Lambda\, g_{\mu\nu} , \qquad \nabla_{\nu}\pi_{\kappa}{}^{\lambda\mu\nu} = 0 . \tag{21}$$

Using these equations, the example Lagrangian $\mathcal{L}_{\mathrm{A}}$ rewritten to the Palatini picture has the following form:

$$\mathcal{L}_{\mathrm{P}} = \frac{\sqrt{|\det g|}}{16\pi} K_{\mu\nu}\, g^{\mu\nu} - \frac{\sqrt{|\det g|}\,\Lambda}{8\pi}\,,$$

or in the metric picture:

$$\mathcal{L}_g = \frac{\sqrt{|\det g|}}{16\pi} \overset{\circ}{R} - \frac{\sqrt{|\det g|}\,\Lambda}{8\pi}\,.$$

This example has one more important property, namely, all three Lagrangians $\mathcal{L}_{\mathrm{A}}$, $\mathcal{L}_{\mathrm{P}}$, $\mathcal{L}_g$ are equivalent *on shell* but also numerically equal. It holds because the field equations (21) guarantee the metricity of the connection.

5. Generalisation of the metric picture

The last variational formula, and somehow the topic of this paper, is the generalised metric picture which will be constructed from the Palatini picture *on shell*. It automatically guarantees the equivalence but also completes the variational description.

As it was written in Section 3, the Palatini Lagrangian has an analogous structure to the metric Lagrangian — it is a sum of Hilbert and matter Lagrangians (of course, they are not equal to those in the metric picture!). Naively, one can assume that inserting metric connection $\overset{\circ}{\Gamma}$ into $\mathcal{L}_{\mathrm{P}}$ gives $\mathcal{L}_g$. Indeed, it is a necessary step to use the field equation (17), but unfortunately, it is not sufficient. Obviously, equation (17) could be impossible to solve analytically, then the solution could be treated in the following way:

$$N^{\kappa}{}_{\lambda\mu} := \Gamma^{\kappa}{}_{\lambda\mu} - \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu}\,, \tag{22}$$

where N is the non-metricity tensor and $\overset{\circ}{\Gamma}$ is the metric (Levi-Civita) connection. Using the variation of Palatini Lagrangian (15) *on shell* and the decomposition of the general affine connection (22), it is obvious that

$$\delta\mathcal{L}_{\mathrm{P}} \;=\; \partial_\nu \left(p^\nu\, \delta\phi + \pi_\kappa{}^{\lambda\mu\nu}\, \delta N^{\kappa}{}_{\lambda\mu} \right) + \partial_\nu \left(\pi_\kappa{}^{\lambda\mu\nu}\, \delta \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu} \right)\,.$$

The last term is a part of Hilbert variation (1), so

$$\delta\mathcal{L}_{\mathrm{P}} \;=\; \delta\mathcal{L}_{\mathrm{H}} + \frac{1}{16\pi}\, \overset{\circ}{\mathcal{G}}{}^{\mu\nu}\, \delta g_{\mu\nu} + \partial_\nu \left(p^\nu\, \delta\phi + \pi_\kappa{}^{\lambda\mu\nu}\, \delta N^{\kappa}{}_{\lambda\mu} \right)\,.$$

It could be proved that

$$\partial_\nu \left(\pi_\kappa{}^{\lambda\mu\nu}\, \delta N^{\kappa}{}_{\lambda\mu} \right) = \partial_\kappa \left(\mathcal{R}^{\mu\nu\kappa}\, \delta g_{\mu\nu} \right) - \delta \left[\overset{\circ}{\nabla}_\kappa \mathcal{R}_\sigma{}^{\sigma\kappa} \right]\,, \tag{23}$$

where

$$\mathcal{R}^{\mu\nu\kappa} := \frac{\sqrt{|\det g|}}{16\pi}\left[N^{\kappa\mu\nu} - N_{\sigma}{}^{\sigma(\mu}\, g^{\nu)\kappa} + \frac{1}{2}\left(N^{\kappa\sigma}{}_{\sigma} - N_{\sigma}{}^{\sigma\kappa}\right) g^{\mu\nu}\right]. \tag{24}$$

The above definition shows the equivalence between $\mathcal{R}$ and N because this equation could be easily inverted to the function $N(\mathcal{R})$. Then

$$\partial_\kappa \left(\mathcal{R}^{\mu\nu\kappa}\, \delta g_{\mu\nu}\right) = \mathcal{Y}^{\lambda\mu}_{\ \ \kappa}\, \delta \mathring{\Gamma}^{\kappa}{}_{\lambda\mu} + \left(\mathring{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa}\right) \delta g_{\mu\nu}\,,$$

where

$$\mathcal{Y}^{\lambda\mu}_{\ \ \kappa} := \mathcal{R}_{\kappa}{}^{\lambda\mu} + \mathcal{R}_{\kappa}{}^{\mu\lambda}\,. \tag{25}$$

Due to the above calculations, the last variation of the Palatini picture is

$$\begin{aligned}\delta\mathcal{L}_{\mathrm{P}} \;&=\; \delta\mathcal{L}_{\mathrm{H}} - \delta\left[\mathring{\nabla}_\kappa \mathcal{R}_{\sigma}{}^{\sigma\kappa}\right] + \partial_\nu\left(p^\nu\, \delta\phi\right)\\ &\quad + \left(\frac{1}{16\pi}\,\mathring{\mathcal{G}}^{\mu\nu} + \mathring{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa}\right)\delta g_{\mu\nu} + \mathcal{Y}^{\lambda\mu}_{\ \ \kappa}\, \delta\mathring{\Gamma}^{\kappa}{}_{\lambda\mu}\,.\end{aligned}$$

Finally, the proper metric Lagrangian is defined as $\mathcal{L}_g := \mathcal{L}_{\mathrm{P}} + \mathring{\nabla}\mathcal{R}$. Analogously, the matter Lagrangian is $\mathcal{L}_{\mathrm{matt}} := \mathcal{L}_g - \mathcal{L}_{\mathrm{H}}$. The variation of $\mathcal{L}_{\mathrm{matt}}$ is the following:

$$\delta\mathcal{L}_{\mathrm{matt}} = \partial_\nu\left(p^\nu\, \delta\phi\right) + \left(\frac{1}{16\pi}\,\mathring{\mathcal{G}}^{\mu\nu} + \mathring{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa}\right)\delta g_{\mu\nu} + \mathcal{Y}^{\lambda\mu}_{\ \ \kappa}\, \delta\mathring{\Gamma}^{\kappa}{}_{\lambda\mu}\,, \tag{26}$$

which will imply the proper Einstein equation

$$\frac{1}{16\pi}\,\mathring{\mathcal{G}}^{\mu\nu} + \mathring{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa} = \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}}\,. \tag{27}$$

Interestingly, putting the second term on the right-hand side of above equality provides the so-called variational derivative of matter Lagrangian over the metric tensor

$$\frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}} - \mathring{\nabla}_\kappa \mathcal{R}^{\mu\nu\kappa} = \frac{\delta\mathcal{L}_{\mathrm{matt}}}{\delta g_{\mu\nu}} := \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu}} - \partial_\kappa \frac{\partial\mathcal{L}_{\mathrm{matt}}}{\partial g_{\mu\nu,\kappa}}\,.$$

Analogously to the metric picture, this quantity could be called a symmetric tensor density (8) but the time–time component of it may not correlate with the energy of such matter.

The last conclusion which is made refers to the relation between matter and non-metricity of the connection. Due to the fact that $\mathcal{L}_{\mathrm{matt}}$ depends on $\overset{\circ}{\nabla}\phi$, the derivative of such Lagrangian over the metric connection $\overset{\circ}{\Gamma}$ does not vanish

$$\frac{\partial \mathcal{L}_{\mathrm{matt}}}{\partial \overset{\circ}{\Gamma}{}^{\kappa}{}_{\lambda\mu}} = \mathcal{Y}^{\lambda\mu}{}_{\kappa} \neq 0 \, .$$

On the other hand, this quantity is related to $\mathcal{R}^{\mu\nu\kappa}$ (25) and then to the non-metricity tensor $N^{\kappa}{}_{\lambda\mu}$ (24). It clearly shows that theories described by the covariant derivatives could be written in much easier (and somehow more natural) non-metric description. On the other hand, the theories with non-metric connection can be naturally translated to metric theories with matter fields [1–5, 7, 8].

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