HAMILTONIAN CHARGES IN SPACETIMES WITH A POSITIVE COSMOLOGICAL CONSTANT*

P.T. Chruściel

Faculty of Physics, University of Vienna, Austria

T. Smołka

Department of Mathematical Methods in Physics, Faculty of Physics
University of Warsaw, Warszawa, Poland

Received 30 November 2021, accepted 3 December 2021, published online 31 January 2022

We analyse the Hamiltonian charges for Maxwell and scalar field theory on light-cones on a de Sitter background.

DOI:10.5506/APhysPolBSupp.15.1-A11

1. Introduction

The recent gravitational waves detections, together with the measurement of a positive cosmological constant Λ, lead to an urgent need for an understanding of gravitational waves in spacetimes with $\Lambda > 0$. In particular, there is a need to understand the Hamiltonian charges, which contain global information about the fields.

In a series of papers, Ashtekar *et al.* (*cf.* [1] and references therein) proposed a framework to study the linearized theory, including a quadrupole formula for the flux of energy flux on a de Sitter background. Not unexpectedly, a positive cosmological constant $\Lambda > 0$ allows for negative energy carried away by gravitational waves, though the authors argued that physically reasonable sources would radiate waves of positive energy. Another approach to this problem, by Chruściel and Ifsits, isolates the *renormalised light-cone volume* [2] as one of the ingredients of a physically meaningful definition.

Our further studies of this problem [3, 4] showed the need for renormalisation of the Hamiltonian energy even if weak (linearised) gravitational fields on a de Sitter background are considered. The question arises about the

* Presented at the 7$^{\text{th}}$ Conference of the Polish Society on Relativity, Łódź, Poland, 20–23 September 2021.

status of the remaining Hamiltonian charges (compare [5]). As a warm-up for the gravitational problem, we undertook an analysis of these charges for Maxwell electromagnetism and scalar field theory on light-cones in a de Sitter background. Here we report on some of these results.

2. Energy and angular momentum in Maxwell theory

We apply the formalism developed in [4] to Maxwell fields on Minkowski, de Sitter, and anti-de Sitter spacetimes. We consider simultaneously all three spacetimes in Bondi coordinates (u, r, x^A). In these, the metric takes the form of

$$g \equiv g_{\alpha\beta}\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \epsilon N^2\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + r^2\underbrace{\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)}_{=:\mathring{\gamma}}, \tag{1}$$

where $N := \sqrt{|(1-\alpha^2r^2)|}$, $\alpha \in \{0, \sqrt{\frac{\Lambda}{3}}\} \subset \mathbb{R}\cup i\mathbb{R}$, $\epsilon \in \{\pm 1\}$, with ϵ equal to one if $1-\alpha^2r^2 < 0$, and minus one otherwise; note that any $\Lambda \in \mathbb{R}$ is allowed, and hence $\alpha \in \mathbb{C}0$ but $\alpha^2 \in \mathbb{R}$. We also define $\mathring{\gamma}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$, while $\mathring{D}_A$ denotes the Levi-Civita derivative associated with $\mathring{\gamma}$.

Conformal invariance of the sourceless Maxwell equations,

$$\nabla_\mu F^{\mu\nu} = 0\,, \qquad \nabla_{[\mu}F_{\kappa\lambda]} = 0\,, \tag{2}$$

shows that for solutions that evolve out of smooth initial data on a spacelike Cauchy surface in de Sitter spacetime, the $(u, x := 1/r, x^A)$ components of the Maxwell field are smooth functions of (u, x, x^A). In $(u, x := 1/r, x^A)$ coordinates, this translates to

$$F = F_{xu}\mathrm{d}x\wedge\mathrm{d}u + F_{xA}\mathrm{d}x\wedge\mathrm{d}x^A + F_{uA}\mathrm{d}u\wedge\mathrm{d}x^A + \frac{1}{2}F_{AB}\mathrm{d}x^A\wedge\mathrm{d}x^B\,, \tag{3}$$

with F_{xu}, *etc.*, having full Taylor expansions in $x \equiv 1/r$ around $x = 0$. In particular, the fields F_{Ar} have expansions of the form of

$$F_{Ar} = -r^{-2}F_{xu} = \overset{(-2)}{F}{}_{Ar}r^{-2} + \ldots\,, \tag{4}$$

where the expansion coefficients are functions of u and x^A. The vacuum Maxwell equations provide the asymptotics of the remaining components of Maxwell tensor, as discussed in detail in [6].

The Noether-type currents for Maxwell fields are obtained from the Lagrangian, which in our signature reads

$$\mathcal{L}\left(A_\mu, \partial A_\mu\right) = -\frac{1}{16\pi}\sqrt{|-\det g|}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\,. \tag{5}$$

We write $A_{\mu,\nu}$ for $\partial_\nu A_\mu$. The Hamiltonian current is defined as

$$\begin{aligned}\mathcal{H}^\mu[X] &:= \frac{\partial\mathcal{L}}{\partial A_{\beta,\mu}}\boldsymbol{L}_X A_\beta - \mathcal{L}X^\mu \\ &= -\frac{1}{4\pi}\sqrt{|-\det g|}\left(F^{\mu\beta}\boldsymbol{L}_X A_\beta - \frac{1}{4}\left(F^{\nu\beta}F_{\nu\beta}\right)X^\mu\right), \qquad (6)\end{aligned}$$

where the usual Lie derivative for tensors has been replaced[1] by $\boldsymbol{L}_X A_\mu := X^\nu F_{\nu\mu}$ to ensure gauge-independence of the Hamiltonian current (6).

If we assume that X is a Killing vector field and the Maxwell field is sourceless, then the Lie derivative of the Hamiltonian current (6) reads

$$\mathcal{L}_Y\mathcal{H}^\mu[X] = -\frac{\sqrt{|-\det g|}}{2\pi}\nabla_\sigma\left[Y^{[\sigma}F^{\mu]\alpha}X^\kappa F_{\kappa\alpha} - \frac{1}{4}Y^{[\sigma}X^{\mu]}F^{\alpha\beta}F_{\alpha\beta}\right]. \quad (7)$$

As in [4], we denote by C_u the light cone of constant retarded time u. Somewhat to our surprise, in contradistinction with the scalar field and with the linearised gravitational field, one finds that the energy and angular momentum integrals over C_u are convergent. The most interesting charge is the energy-like integral associated with the motion of the tip of the light cone to the future along the flow of a Killing vector[2] $\mathcal{T} \equiv \partial_u$. Letting $\mathrm{d}\mu_C = \sqrt{\det g_{AB}}\,\mathrm{d}r\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3$ and $\mathrm{d}\mu_{\mathring{\gamma}} = \sqrt{\det\mathring{\gamma}_{AB}}\,\mathrm{d}x^2\wedge\mathrm{d}x^3$, energy is defined as $E[C_u,\mathcal{T}] := \int_{C_u}\mathcal{H}^\mu[\partial_u]\mathrm{d}S_\mu = \int_{C_u}\mathcal{H}^u[\partial_u]\mathrm{d}\mu_C$. We find

$$E\left[C_u,\mathcal{T}\right] = \frac{1}{16\pi}\int\limits_{C_u}\left(\frac{1}{r^2}\mathring{\gamma}^{AC}\mathring{\gamma}^{BD}F_{AB}F_{CD} + 2F_{ur}^2 - 2\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\right)\mathrm{d}r\,\mathrm{d}\mu_{\mathring{\gamma}}\,. \quad (8)$$

The total angular-momentum, defined as $J[\mathcal{R}] := \int_{C_u}\mathcal{H}^\mu[\mathcal{R}]dS_\mu$, is generated by a Killing field $\mathcal{R} = \varepsilon^{BA}\mathring{D}_A\big(R_i n^i\big)\partial_B$, where R_i are constants. One finds

$$J[\mathcal{R}] = \frac{1}{4\pi}R_i\int\limits_{C_u}\varepsilon^{AB}\mathring{D}_B n^i\left(r^2F_{ur}F_{Ar} + \mathring{\gamma}^{BC}F_{Br}F_{CA}\right)\mathrm{d}r\,\mathrm{d}\mu_{\mathring{\gamma}}\,. \quad (9)$$

Key physical information is provided by the rate of change of the energy and the angular momentum as the tip of the light cone is moved to the

[1] $\boldsymbol{L}_X A_\mu$ is a natural definition for the Lie derivative of a connection one form on a U(1) principal bundle.

[2] Recall that ∂_u is timelike at the tip of the light cone so that each subsequent cone so obtained lies to the future of the preceding one.

future along the flow of the Killing vector $\mathcal{T} := \partial_u$. Using (7), we find

$$\begin{aligned}\frac{\mathrm{d}E[C_u]}{\mathrm{d}u} &= -\lim_{R\to\infty}\frac{1}{4\pi}\int\limits_{S_R}\left[r^2F_{ur}^2+\mathring{\gamma}^{AB}F_{uA}F_{rB}+\epsilon N^2\mathring{\gamma}^{AB}F_{rA}F_{rB}\right]_{r=R}\mathrm{d}\mu_{\mathring{\gamma}}\\ &= -\frac{1}{4\pi}\int\limits_{S_\infty}\left[\mathring{\gamma}^{AB}\left(\alpha^2\overset{(-2)}{F}{}_{Ar}\overset{(0)}{F}{}_{Bu}+\overset{(0)}{F}{}_{Au}\overset{(0)}{F}{}_{Bu}\right)\right]\mathrm{d}\mu_{\mathring{\gamma}}\,. \end{aligned} \tag{10}$$

Analogously, the u-derivative of angular momentum is given by

$$\begin{aligned}\frac{\mathrm{d}J[\mathcal{R},C_{u,R}]}{\mathrm{d}u} &:= -\frac{1}{4\pi}R_i\lim_{R\to\infty}\int\limits_{S_R}\varepsilon^{AB}\mathring{D}_Bn^i\Big[r^2F_{ur}F_{uA}-\epsilon N^2\mathring{\gamma}^{BC}F_{rB}F_{CA}\\ &\qquad -\mathring{\gamma}^{BC}F_{uB}F_{CA}\Big]_{r=R}\mathrm{d}\mu_{\mathring{\gamma}}\\ &= -\frac{1}{4\pi}R_i\int\limits_{S_\infty}\left[\varepsilon^{AB}\mathring{D}_B\left(n^i\right)\left(\mathring{\gamma}^{BC}\left(\alpha^2\overset{(-2)}{F}{}_{Br}+\overset{(0)}{F}{}_{Bu}\right)\overset{(0)}{F}{}_{CA}\right.\right.\\ &\qquad \left.\left.-\overset{(-2)}{F}{}_{ur}\overset{(0)}{F}{}_{Au}\right)\right]\mathrm{d}\mu_{\mathring{\gamma}}\,. \end{aligned} \tag{11}$$

3. Energy and angular momentum in scalar field theory

In [4], we found that the canonical energy on light cones for a natural class of linear scalar fields in de Sitter spacetime was generically infinite, and had to be renormalised. Here, we consider convergence of the integrals defining the total angular momentum. We recall the results for energy from [4].

In our signature, the Lagrangian density reads

$$\mathcal{L} = -\frac{1}{2}\sqrt{|-\det g|}\left(g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi+m^2\phi^2\right)\,, \tag{12}$$

and we choose $m = 2\alpha^2$ to obtain a conformally-invariant field equation

$$\Box_g\phi-\underbrace{\frac{(d-2)R(g)}{4(d-1)}}_{=:m^2}\phi = 0\,, \tag{13}$$

where d is the dimension of spacetime ($d = 4$ in our case) and $R(g)$ is the scalar curvature of g.

The canonical energy-momentum current $\mathcal{H}^\mu$ equals

$$\mathcal{H}^\mu[X] = -\sqrt{|-\det g|}\left(\nabla^\mu\phi\,\mathcal{L}_X\phi-\frac{1}{2}\left(\nabla^\alpha\phi\nabla_\alpha\phi+m^2\phi^2\right)X^\mu\right)\,. \tag{14}$$

As for the Maxwell field, we consider simultaneously Minkowski spacetime and (anti-)de Sitter spacetime in coordinates as in (1).

If X is a Killing vector and the Klein–Gordon equation of motion (13) is assumed, then the Lie derivative of the Hamiltonian (6) reads

$$-\frac{\mathcal{L}_Y \mathcal{H}^\mu[X]}{\sqrt{|-\det g|}} = 2\nabla_\sigma \left(Y^{[\sigma}\nabla^{\mu]}\phi X^\alpha \nabla_\alpha \phi - \frac{1}{2} Y^{[\sigma} X^{\mu]} \left(\nabla^\alpha\phi\nabla_\alpha\phi + m^2\phi^2\right)\right) . \tag{15}$$

It follows from [4] (*cf.* Section 2.2.1 there) that scalar fields evolving out of smooth initial data on a Cauchy surface have an asymptotic expansion of the form of

$$\phi(u,r,x^A) = \frac{\overset{(-1)}{\phi}(u,x^A)}{r} + \frac{\overset{(-2)}{\phi}(u,x^A)}{r^2} + \frac{\overset{(-3)}{\phi}(u,x^A)}{r^3} + \ldots . \tag{16}$$

We denote by $C_{u,R} = C_u \cap \{r \le R\}$ the truncation of the light cone to $r = R$. We use the asymptotics (16) which applies both to the $\alpha = 0$ case with $m = 0$ and to the case $\alpha^2 = m^2/2$ with $m \neq 0$. The energy on the truncated cone is defined as $E[\mathcal{T}, C_{u,R}] := \int_{C_{u,R}} \mathcal{H}^\mu[\partial_u] \mathrm{d}S_\mu = \int_{C_{u,R}} \mathcal{H}^u[\partial_u]\mathrm{d}S_u$, and reads

$$\begin{aligned} E[\mathcal{T}, C_{u,R}] &= \frac{1}{2} \int\limits_{C_{u,R}} \left(\mathring{\gamma}^{AB} \mathring{D}_A\phi \mathring{D}_B\phi + m^2 r^2 \phi^2 + \left(r^2 - \alpha^2 r^4\right)\left(\partial_r\phi\right)^2 \right) \mathrm{d}r\, \mathrm{d}\mu_{\mathring{\gamma}} \\ &= \frac{\alpha^2 R}{2} \int\limits_{S_R} \left(\overset{(-1)}{\phi}\right)^2 \mathrm{d}\mu_{\mathring{\gamma}} + \int\limits_{C_{u,R}} O\left(r^{-2}\right) \mathrm{d}r\, \mathrm{d}\mu_{\mathring{\gamma}} \end{aligned} \tag{17}$$

which, as already mentioned, diverges as R tends to infinity; a renormalised finite energy is obtained by dropping the boundary integral in (17).

As for the Maxwell field, the total angular-momentum is obtained from the integral $J[\mathcal{R}, C_{u,R}] := \int_{C_{u,R}} \mathcal{H}^\mu[\mathcal{R}]\mathrm{d}S_\mu \equiv R_i J^i[C_{u,R}]$, where the J^is are given by

$$J^i[C_{u,R}] := \int\limits_{C_{u,R}} r^2 \varepsilon^{AB} \mathring{D}_B n^i \mathring{D}_A \phi \partial_r \phi \, \mathrm{d}r\, \mathrm{d}\mu_{\mathring{\gamma}} = \int\limits_{C_{u,R}} O\left(r^{-2}\right) \mathrm{d}r\, \mathrm{d}\mu_{\mathring{\gamma}} \,, \tag{18}$$

so that the limit $R \to \infty$ is finite, even though the total energy diverges.

As before, a question of interest is the rate of change of the charge integrals as the tip of the light cone is moved to the future along a flow of the Killing vector $\partial_u \equiv \mathcal{T}$. Using (15), we obtain the energy-flux

$$\frac{\mathrm{d}E\left[\mathcal{T}, C_{u,R}\right]}{\mathrm{d}u} = -\int\limits_{S_R} \left[r^2\left(\partial_u\phi + \left(\alpha^2 r^2 - 1\right)\partial_r\phi\right)\partial_u\phi\right]_{r=R} \mathrm{d}\mu_{\mathring{\gamma}}$$
$$= \int\limits_{S_R} \left[\alpha^2 \overset{(-1)}{\phi}\,\partial_u \overset{(-1)}{\phi} R + \alpha^2 \overset{(-1)}{\phi}\,\partial_u \overset{(-2)}{\phi} + \left(2\alpha^2 \overset{(-2)}{\phi} - \partial_u \overset{(-1)}{\phi}\right)\partial_u \overset{(-1)}{\phi} + O\left(\frac{1}{R}\right)\right] \mathrm{d}\mu_{\mathring{\gamma}}\,. \tag{19}$$

The u-derivative of angular momentum is given by

$$\frac{\mathrm{d}J\left[\mathcal{R}, C_{u,R}\right]}{\mathrm{d}u} = -R^i \int\limits_{S_R} \varepsilon^{AB}\mathring{D}_B n^i \left[r^2\left(\partial_u\phi + \left(\alpha^2 r^2 - 1\right)\partial_r\phi\right)\left(\mathring{D}_A\phi\right)\right]_{r=R} \mathrm{d}\mu_{\mathring{\gamma}}$$
$$= R^i \int\limits_{S_R} \left[\varepsilon^{AB}\mathring{D}_B\left(n^i\right)\left(\alpha^2 \overset{(-1)}{\phi}\,\mathring{D}_A \overset{(-2)}{\phi} + \left(2\alpha^2 \overset{(-2)}{\phi} - \partial_u \overset{(-1)}{\phi}\right)\mathring{D}_A \overset{(-1)}{\phi}\right)\right] \mathrm{d}\mu_{\mathring{\gamma}}\,. \tag{20}$$

4. Summary

We have analysed the total energy and angular momentum of the Maxwell field and of a conformally-invariant scalar field on de Sitter and Minkowski backgrounds. Energy for linearised gravity in a similar setup has been analysed in our previous works [3, 4]. Somewhat surprisingly, Maxwell fields have finite total charges, as opposed to linearised gravitational fields and scalar fields. A detailed analysis of all Hamiltonian charges[3], including an investigation of the algebra of charges, will be presented in [6].

REFERENCES

[1] A. Ashtekar, B. Bonga, A. Kesavan, «Asymptotics with a positive cosmological constant. III. The quadrupole formula», *Phys. Rev. D* **92**, 104032 (2015), arXiv:1510.05593 [gr-qc].

[2] P. Chruściel, L. Ifsits, «The cosmological constant and the energy of gravitational radiation», *Phys. Rev. D* **93**, 124075 (2016), arXiv:1603.07018 [gr-qc].

[3] P. Chruściel, S.J. Hoque, T. Smołka, «Energy of weak gravitational waves in spacetimes with a positive cosmological constant», *Phys. Rev. D* **103**, 064008 (2021), arXiv:2003.09548 [gr-qc].

[4] P.T. Chruściel, Sk.J. Hoque, M. Maliborski, T. Smołka, «On the canonical energy of weak gravitational fields with a cosmological constant $\Lambda \in \mathbb{R}$», *Eur. Phys. J. C* **81**, 696 (2021), arXiv:2103.05982 [gr-qc].

[5] M. Kolanowski, J. Lewandowski, «Hamiltonian charges in the asymptotically de Sitter spacetimes», *J. High Energy Phys.* **2105**, 063 (2021), arXiv:2103.14674 [gr-qc].

[6] P.T. Chruściel, T. Smołka, in preparation.

[3] We mean all charges associated with ten-dimensional space of Killing vectors for the (anti-)de Sitter space.