SPACE-TIMES WITH ALL PENROSE LIMITS DIAGONALISABLE *

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In this talk, I gave an account of my article `arXiv:1909.07756[gr-qc]` which considered the following question: Penrose gave a construction which associates a plane-wave space-time $P(M,\Gamma)$ with any pair (M,Γ) where M is a space-time and Γ is a null geodesic in M; what condition must M satisfy if P is diagonalisable for every Γ in M?

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1. The problem

It is well-known that, given a smooth 3-dimensional Riemannian or Lorentzian metric, there always locally exist coordinates in which the metric is diagonal, that is to say, the off-diagonal components are all zero (see [1] for the Riemannian case and [2] for the Lorentzian). In 4 dimensions and above this is not true, see *e.g.* [3] or [4]: there are 4-metrics which are not diagonalisable, and in particular for plane waves one can give a necessary and sufficient condition for diagonalisability, which we will recall below.

In [5], Penrose gave a construction which associates a plane-wave space-time $P(M,\Gamma)$ with any pair (M,Γ) where M is a space-time and Γ is a null geodesic in M. The question therefore naturally arises of whether or not, for a given M, every such $P(M,\Gamma)$ can be diagonalised. To approach this question, we first recall the Penrose construction, which can be phrased as follows: in the Brinkman form, a plane-wave metric can be written

$$g = 2\mathrm{d}u\left(\mathrm{d}v + H\left(u,\zeta,\bar{\zeta}\right)\mathrm{d}u\right) - 2\mathrm{d}\zeta\mathrm{d}\bar{\zeta}\,, \tag{1}$$

with

$$H\left(u,\zeta,\bar{\zeta}\right) = \frac{1}{2}\left(\Psi(u)\zeta^2 + 2\Phi(u)\zeta\bar{\zeta} + \bar{\Psi}(u)\bar{\zeta}^2\right)\,; \tag{2}$$

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with complex $\Psi(u)$ and real $\Phi(u)$, which are in fact the remaining nonzero components of the Weyl and Ricci spinors respectively, in a suitable spinor dyad. Now, given an arbitrary space-time M and a null geodesic Γ in M, one may choose a spinor α^A along Γ so that the null tangent to Γ is $\ell^a = \alpha^A\bar{\alpha}^{A'}$ and

$$D\alpha^A := \ell^b\nabla_b\alpha^A = 0\,,$$

and an affine parameter u along Γ so that $Du = 1$. Then one calculates the curvature components

$$\Psi(u) := \psi_{ABCD}\alpha^A\alpha^B\alpha^C\alpha^D\,, \qquad \Phi(u) := \phi_{ABA'B'}\alpha^A\alpha^B\bar{\alpha}^{A'}\bar{\alpha}^{B'} \tag{3}$$

along Γ, where $\psi_{ABCD}, \phi_{ABA'B'}$ are respectively the Weyl and Ricci spinors of M, then substitutes these into H in (2) and this into g in (1) — this gives the metric of $P(M,\Gamma)$. Penrose defined $P(M,\Gamma)$ via a genuine limiting process but the result is the same. There is a freedom to multiply α^A by a complex constant, say λ, but this changes $P(M,\Gamma)$ only by a diffeomorphism.

We recall that it was shown in [3] that the plane-wave metric (1) is diagonalisable iff the phase of $\Psi(u)$ is constant in u, a condition which can be written

$$\bar{\Psi}\dot{\Psi} - \Psi\dot{\bar{\Psi}} = 0\,, \tag{4}$$

with the dot for $\mathrm{d}/\mathrm{d}u$.

Now we can pose the problem: what is the condition on M if $P(M,\Gamma)$ is diagonalisable for every Γ?

2. The solution

If $P(M,\Gamma)$ is diagonalisable for every choice of Γ in a given M, then (4) must hold with (3) and every choice of α^A, which is only possible if

$$\bar{\psi}_{(A'B'C'D'}\nabla_{E')(A}\psi_{BCDE)} - \psi_{(ABCD}\nabla_{E)(A'}\bar{\psi}_{B'C'D'E')} = 0\,. \tag{5}$$

This is clearly a strong condition on M — twenty-one real conditions on the ten real components of the Weyl tensor. To analyse it, suppose first that the scalar invariant

$$I := \psi_{ABCD}\psi^{ABCD}$$

is nonzero, then contract (5) with $\bar{\psi}^{A'B'C'D'}$ to obtain

$$\frac{3\bar{I}}{5}\nabla_{E'(A}\psi_{BCDE)} = \psi_{(ABCD}W_{E)E'}$$

with

$$W_{EE'} = \bar{\psi}^{A'B'C'D'}\nabla_{E(E'}\bar{\psi}_{A'B'C'D')}\,.$$

Set $U_{EE'} := \frac{5}{3I} W_{EE'}$ so that

$$\nabla_{E'(A}\psi_{BCDE)} = \psi_{(ABCD}U_{E)E'}\,, \tag{6}$$

and substitute back into (5) to see that $U_{EE'}$ must necessarily be a real vector. Now calculate a contracted derivative of (6):

$$\nabla_{E'(A}\nabla^{E'}_{B}\psi_{CDEF)} = \nabla_{E'(A}\left(U^{E'}_{B}\psi_{CDEF)}\right)\,,$$

expand both sides, and use the Ricci identities and (6) to deduce that

$$\nabla_{E'(A}U^{E'}_{B)} = 0\,.$$

Since $U_{AA'}$ is real, this means that it is closed and therefore exact, say

$$U_{AA'} = \nabla_{AA'}U\,,$$

for some real function U, when finally (6) implies that $\omega_{ABCD} := \mathrm{e}^{-U}\psi_{ABCD}$ is a valence-4 Killing spinor

$$\nabla_{A'(A}\omega_{BCDE)} = 0\,.$$

It is known that if M admits a valence-4 Killing spinor then it must be proportional to the Weyl spinor, but this is a stronger condition since the function of proportionality must be real.

If $I = 0$, then the calculation above goes through using the other scalar invariant $J = \psi_{ABCD}\psi^{AB}{}_{PQ}\psi^{CDPQ}$, and in the degenerate cases, when I and J are both zero (these have algebraically special Weyl spinor), one has to proceed type by type in the Petrov–Pirani–Penrose (or PPP) classification. In all cases, the conclusion is the same: the Weyl spinor must be a real function times a valence-4 Killing spinor.

One may now seek to classify space-times with such a Weyl spinor. Considering vacuum or Einstein metrics first, a useful observation is that every principal spinor of a Killing spinor is geodesic and shear-free [6], but for vacuum or Einstein metrics principal spinors of the Weyl spinor are only geodesic and shear-free if they are repeated (see *e.g.* [7]). Thus the only vacuum cases to try are type N and type D in the PPP classification, and in the first, the Weyl spinor needs to be proportional with a real factor to the fourth power of a valence-1 Killing spinor (and these metrics are known — see [8] for references), while in the second, it is proportional with a real factor to the square of a valence-2 Killing spinor (and again these are known — see [8]). Among the type D examples, it is easy to see that the Schwarzschild metric has the desired property — so all Penrose limits of the Schwarzschild metric are diagonalisable — but the Kerr metric with $a \neq 0$ does not — the Weyl spinor for Kerr is a *complex* function times a valence-4 Killing spinor rather than a real one. In the algebraically general case, there can be no vacuum examples but there are interesting nonvacuum ones. Here, the Killing spinor, and therefore also the Weyl spinor has four distinct principal spinors so that there are four geodesic and shear-free congruences. Precisely, this case but in Riemannian signature was investigated by Kobak [9]. He considered the metrics

$$g = \mathrm{d}z\mathrm{d}\bar{z} + f\mathrm{d}w\mathrm{d}\bar{w}$$

with f real and $f(z+\bar{w}, \bar{z}+w)$. These metrics by design have two distinct Hermitian structures: one has $(\mathrm{d}z, \mathrm{d}w)$ as holomorphic one-forms and the other has $(\mathrm{d}z - f\mathrm{d}\bar{w}, \mathrm{d}w + \mathrm{d}\bar{z})$[1]. If they could be made Lorentzian, then each Hermitian structure would give rise to two geodesic and shear-free congruences, giving the desired four such congruences. If we set $z = x + iy$ and $w = u + iv$, restrict f to $f = f(x+u)$, and then set $y = it$ and switch the sign on the metric, it becomes real and Lorentzian

$$g = \mathrm{d}t^2 - \mathrm{d}x^2 - f(x+u)\left(\mathrm{d}u^2 + \mathrm{d}v^2\right) ,$$

and it can be checked [8] that the Weyl spinor is algebraically general and proportional to a Killing spinor with a real function of proportionality. Thus all its Penrose limits are diagonalisable. (The above and Schwarzschild are two examples of diagonal metrics with all Penrose limits diagonalisable but, as was seen in [8], the Kasner metric is, in general, a diagonal metric with some Penrose limits diagonalisable but most non-diagonalisable.)

3. An afterthought

In [8], I calculated the Penrose limits of the Kasner metric, which is

$$g = \mathrm{d}t^2 - t^{2p}\mathrm{d}x^2 - t^{2q}\mathrm{d}y^2 - t^{2r}\mathrm{d}z^2 ,$$

where p, q, r are real constants. This process begins with solving the null geodesic equations, which are the Euler–Lagrange equations for the Lagrangian

$$L := \tfrac{1}{2}\left(\dot{t}^2 - t^{2p}\dot{x}^2 - t^{2q}\dot{y}^2 - t^{2r}\dot{z}^2\right) ,$$

together with $L = 0$. The Kasner metric has three obvious Killing vectors, ∂_x, ∂_y and ∂_z so there are at once three constants of the motion

$$c_1 = t^{2p}\dot{x} , \qquad c_2 = t^{2q}\dot{y} , \qquad c_3 = t^{2r}\dot{z} ,$$

[1] There are sign errors in these in [8].

after which the geodesic equation is solved by quadratures, and one can proceed to find $P(M, \Gamma)$ for different Γ.

My conclusion in [8] was that, in general, the Penrose limit was non-diagonalisable (so that, in particular, a diagonalisable M can have non-diagonalisable $P(M, \Gamma)$), but it would be diagonalisable if the product $c_1 c_2 c_3$ was zero. Stated differently, the Penrose limit would be diagonalisable for any null geodesic Γ lying in one of the 3-surfaces of constant x or constant y or constant z. In discussion after my talk at POTOR7, I realised that the following must be true (see [10]):

Proposition. *Suppose Σ is a time-like* umbilic *hypersurface in a space-time M, and Γ is a null geodesic of M lying entirely in Σ, then the Penrose limit $P(M, \Gamma)$ is diagonalisable.*

To see this, first recall that Σ is umbilic if and only if its second fundamental form is proportional to its metric, and then any null geodesic Γ through a point in Σ and initially tangent to Σ will remain in Σ (Σ is *totally geodesic for null geodesics* and one needs Σ to be time-like in order for there to be any such Γ). Next, one shows from the Codazzi equations that the magnetic part of the Weyl tensor is necessarily zero on an umbilic hypersurface (recall the magnetic part of the Weyl tensor is defined by

$$H_{ab} = \tfrac{1}{2}\epsilon_{ac}{}^{pq} C_{pqbd} N^c N^d \,,$$

where C_{abcd} is the space-time Weyl spinor and N^a is the unit normal to Σ). Finally, one shows from this that $\Psi(u)$ has constant phase along any Γ lying in Σ. Now a 3-surface Σ orthogonal to a hypersurface-orthogonal Killing vector necessarily has vanishing second fundamental form — it is *totally geodesic* and so in particular is umbilic. This explains the observation above about the limits of the Kasner metric. This Proposition also gives a reason why every Penrose limit of the Schwarzschild metric is diagonalisable: with the aid of a rotation any null geodesic of the Schwarzschild metric may be supposed to lie in the equatorial plane, but the equatorial plane is the fixed point set of the involution $\theta \to \pi - \theta$ and so is again totally geodesic and therefore umbilic.

I am grateful for the opportunity to attend POTOR7 and for hospitality received there. I include here in Section 3 a result arising after my talk (see) [10]. For a longer account of the earlier material and fuller references see [8].

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