(IN)COMPLETENESS OF QUASINORMAL MODES*

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A perturbed black hole emits radiation at certain characteristic frequencies, the quasinormal frequencies, similar to the spectrum of frequencies produced by a struck guitar string. The normal modes of a guitar string are complete, in the sense that any oscillation of the string may be written as a superposition of these modes. In the case of quasinormal modes, this is not the case in general. We present here a simple proof of this fact.

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1. Introduction

It is a classical, and well-known, fact that any solution to the wave equation on a bounded interval

$$-\partial_t^2 u + \partial_x^2 u = 0\,, \qquad u(0) = u(\pi) = 0$$

may be written as a sum over harmonics

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{-i\sigma_n t} u_n(x)\,, \qquad u_n(x) := \sin nx\,, \qquad \sigma_n = n \tag{1}$$

for some $a_n \in \mathbb{C}$. The functions u_n and frequencies σ_n are characterised by being solutions to the eigenvalue problem

$$\partial_x^2 u + \sigma^2 u = 0\,, \qquad u(0) = u(\pi) = 0\,.$$

It is also a numerical [1, 2] and increasingly an experimental [3] fact that a perturbed black hole will, at least for some time interval, produce radiation that oscillates and decays at certain fixed discrete complex frequencies, known as the quasinormal frequencies. One may ask whether a result similar to (1) holds in this case. That is, are the quasinormal modes complete.

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2. Schwarzschild–de Sitter

We consider the linear wave equation on a black hole background

$$\Box_g \psi = 0\,, \tag{2}$$

where for concreteness we will assume that g is the Schwarzshild–de Sitter metric

$$g = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2m}{r} - \frac{r^2}{l^2}} + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\,,$$

where m, l are chosen such that the polynomial $r^3 - rl^2 + 2ml^2$ has two positive roots, $r_\mathrm{h}, r_\mathrm{c}$, and we restrict attention to the static patch $(t, r, \theta, \phi) \in \mathbb{R} \times (r_\mathrm{h}, r_\mathrm{c}) \times S^2$.

On the static patch, we introduce the usual tortoise coordinate r_* by

$$r_* = \int_{3m}^{r} \frac{\mathrm{d}r'}{1 - \frac{2m}{r'} - \frac{r'^2}{l^2}}\,.$$

We have $-\infty < r_* < \infty$, with the limit $r_* \to -\infty$ corresponding to the black hole horizon at $r = r_\mathrm{h}$ and $r_* \to \infty$ corresponding to the cosmological horizon at $r = r_\mathrm{c}$. The metric becomes

$$g = \left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right)\left(-\mathrm{d}t^2 + \mathrm{d}r_*^2\right) + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2\right)\,,$$

with r understood to be a function of r_*. In these coordinates, the wave equation takes the form of

$$-\partial_t^2(r\psi) + \partial_{r_*}^2(r\psi) + \left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right)\left[\frac{1}{r^2}\Delta_{S^2}(r\psi) - \left(\frac{2m}{r^3} - \frac{2}{l^2}\right)(r\psi)\right] = 0\,.$$

It has been shown by [4, 5] (see also [6]) that solutions to (2) arising from compactly supported data at $t = 0$ have an asymptotic expansion at late times. For any $\nu > 0$, we may write

$$\begin{aligned}\psi(t, r_*, \theta, \phi) \;&=\; \sum_{\mathrm{Re}(\sigma_k) > -\nu} \sum_{j=1}^{m(k)} a_{j,k} t^j \mathrm{e}^{-i\sigma_k t} \frac{1}{r} w_{j,k}(r_*, \theta, \phi) + O\left(\mathrm{e}^{-\nu t}\right)\,, \\ &\text{as } t \;\to\; \infty\,,\end{aligned} \tag{3}$$

for some constants $a_{j,k} \in \mathbb{C}$. The (complex) frequencies σ_k are determined by solving the equation

$$\tilde{L} w := \partial_{r_*}^2 w + \sigma^2 w + \left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right)\left[\frac{1}{r^2}\Delta_{S^2} w - \left(\frac{2m}{r^3} - \frac{2}{l^2}\right) w\right] = 0\,, \tag{4}$$

subject to the condition that w is *outgoing*

$$w \sim e^{i\sigma|r_*|}\,, \qquad \text{as } |r_*| \to \infty\,. \tag{5}$$

Such solutions exist only when σ belongs to a discrete set of values in the lower half-plane, and we have $w_{0,k} := w$. Each such frequency has a multiplicity $m(k) \in \mathbb{Z}_{\geq 0}$, and the functions $w_{j,k}$ can be determined by iteratively solving

$$\tilde{L} w_{j,k} = w_{j-1,k}\,,$$

subject to the condition that $w_{j,k}$ is outgoing.

The functions $r^{-1} t^j e^{-i\sigma t} w_{j,k}$ are solutions to (2) which oscillate and decay in time. We call such a σ a quasinormal frequency (or resonance) and the corresponding $w_{j,k}$ a quasinormal mode (or resonant state).

We note that while (4), with the boundary condition (5), appears to be an eigenvalue problem, this is not really the case since the outgoing boundary condition requires that w *grows* as $|r_*| \to \infty$. Accordingly, the discreteness of the quasinormal mode spectrum is not *a priori* obvious. This can be resolved by the method of complex scaling [7] or, alternatively with the approach of Vasy [8] (see also [9]). This latter approach permits us to relax the assumption that our initial data be compactly supported to an assumption of smoothness with respect to the coordinates of the analytic extension of the static patch.

3. Incompleteness of the quasinormal modes

One might hope that by sending $\nu \to \infty$ in (3), we can write

$$\psi(t, r_*, \theta, \phi) \stackrel{?}{=} \sum_{\sigma_k} \sum_{j=1}^{m(k)} a_{j,k} t^j e^{-i\sigma_k t} \frac{1}{r} w_{j,k}(r_*, \theta, \phi)\,. \tag{6}$$

However, the expansion (3) is valid as $t \to \infty$, *not* as $\nu \to \infty$. To see why this must be the case, we will construct a solution to (2) which cannot be written as a sum over quasinormal modes.

Consider figure 1. It shows the Penrose diagram of the maximal analytic extension of the Schwarzschild–de Sitter spacetime. The static patch corresponds to the central diamond of the figure, it is bounded to the future by the future black hole horizon $\mathscr{H}_{\mathrm{h}}^+$ and future cosmological horizon $\mathscr{H}_{\mathrm{c}}^+$, and by the corresponding horizons $\mathscr{H}_{\mathrm{h}}^-$, $\mathscr{H}_{\mathrm{c}}^-$ in the past.

We now suppose that we specify non-zero functions $\psi_{\mathrm{h}} : \mathscr{H}_{\mathrm{h}}^+ \to \mathbb{C}$ and $\psi_{\mathrm{c}} : \mathscr{H}_{\mathrm{c}}^+ \to \mathbb{C}$, which are smooth and compactly supported. Their support is indicated in figure 1 by the thicker lines on the horizons. By introducing a spacelike surface which intersects $\mathscr{H}_{\mathrm{h}}^+$ and $\mathscr{H}_{\mathrm{c}}^+$ to the future of the support

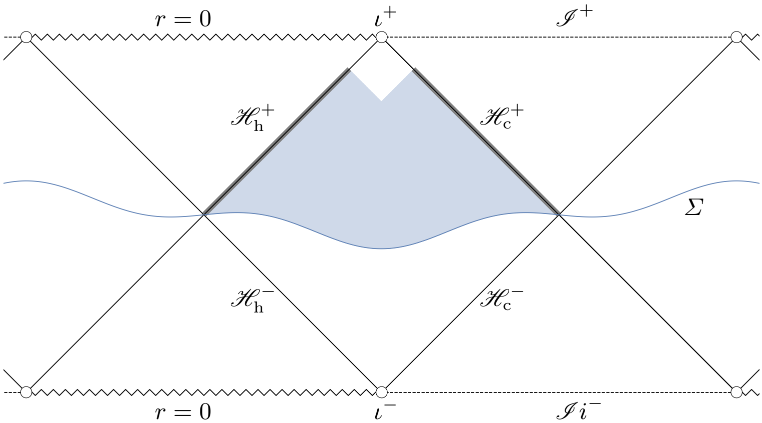

Fig. 1. Construction of the counterexample.

of ψ_h and ψ_c, respectively, and imposing trivial Cauchy data on this surface, we can solve construct a solution to (2) on the static patch such that

$$\psi|_{\mathscr{H}_\mathrm{h}^+} = \psi_\mathrm{h}\,, \qquad \psi|_{\mathscr{H}_\mathrm{c}^+} = \psi_\mathrm{c}\,.$$

This follows from standard results concerning characteristic initial value problems for the linear hyperbolic PDE (we solve 'backwards' in time, *i.e.* down the Penrose diagram). By considering the Cauchy data induced by ψ on a spacelike surface Σ, we can if we wish view ψ as solving a forward evolution problem

Since solutions of the wave equation must respect the causality of the underlying spacetime, the support of the solution to our equation must lie within the shaded region. In other words, for any fixed r_*, there exists $\tau(r_*)$ such that $\psi(t, r_*, \theta, \phi) = 0$ for all $t > \tau(r_*)$. Comparing this with (3), we see that the expansion can only be valid if $a_k = 0$. Thus, if it were the case that (6) holds for this solution, we would deduce that $\psi \equiv 0$. However, this clearly cannot be the case, as by construction, ψ does not vanish on $\mathscr{H}_\mathrm{h}^+$.

We conclude then that the expansion (3) cannot hold in general. In fact, since (2) is linear, we can always add the solution constructed above to any other solution to see that for generic solutions (3) fails.

We should briefly compare this result to those obtained in [10, 11][1]. These papers show that for one-dimensional wave equations with potential, if the initial data is compactly supported, then the solution exactly equals its quasinormal expansion *at late times*. That is (3) holds on regions of bounded r_* for all $t > T$, where T is some constant determined from the

[1] We thank Andrzej Rostworowski for bringing the second of these papers to our attention.

initial data. This is not inconsistent with our discussion above. Indeed, our solution does equal its quasinormal expansion for sufficiently late times. However, it does not equal its quasinormal expansion for *all* times.

4. Conclusion

We have shown that, in contrast to the situation for normal modes on a bounded domain, the quasinormal modes of the Schwarzschild–de Sitter black hole are not complete. In fact, we did not make use of any of the detailed properties of the metric to establish this, so the result holds on much more general black hole spacetimes.

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