

DIRAC PROCEDURE AND THE HAMILTONIAN FORMALISM FOR COSMOLOGICAL PERTURBATIONS IN A BIANCHI I UNIVERSE*

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We apply the Dirac procedure for constrained systems to the Arnowitt–Deser–Misner (ADM) formalism linearized around a Bianchi I universe with a single minimally coupled scalar field. We discuss and employ basic concepts such as Dirac observables, Dirac brackets, gauge-fixing conditions, reduced phase space, physical Hamiltonian, canonical isomorphism between different gauge-fixing surfaces and spacetime reconstruction. We show that the definition of a gravitational wave as a traceless-transverse mode of the metric perturbation needs to be revised. Moreover, there exist coordinate systems in which a polarization mode of the gravitational wave is given entirely in terms of a scalar metric perturbation. The obtained fully canonical formalism will serve as a starting point for a complete quantization of the cosmological perturbations and the cosmological background.

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1. Introduction

From the CMB observations [1], we know that our Universe is mostly isotropic and homogeneous and the present matter structure of our Universe suggests the presence of small initial perturbations. Thus, we work with cosmological perturbation. To study the dynamical origin of initial perturbations, we assume as little primordial symmetries as possible, since symmetries are generally unstable under the backward evolution. For this reason, we drop the assumption of isotropy by working with a Bianchi I metric. We implement the Dirac method for constrained system [2] which allows us to study the Hamiltonian formalism for cosmological perturbations. With this method, the Hamiltonian's degrees of freedom can be easily separated between physical and unphysical ones. The obtained Hamiltonian can also

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be written in a gauge-invariant way by using the Dirac observables, which also give a form suitable for the canonical quantization. We hereby present this approach applied to a perturbed and anisotropic metric to obtain the system's Hamiltonian with the presence of a scalar field ϕ in a potential $V(\phi)$. See [3] for more details.

2. ADM formalism and Hamiltonian

In the ADM formalism [4], the line element reads

$$ds^2 = -N^2 dt^2 + q_{ij} (dx^i + N^i dt) (dx^j + N^j dt) . \quad (1)$$

The phase space includes the 3-metric q_{ij} and the 3-momenta¹ $\pi^{ij} = \sqrt{q} (K^{ij} - K q^{ij})$ which, together with the scalar field ϕ and its momentum π_ϕ , give the canonical pairs satisfying the following canonical relations:

$$\{q_{ij}(x), \pi^{kl}(x')\} = \delta_{(i}^{(k} \delta_{j)}^{l)} \delta^3(x-x'), \quad \{\phi(x), \pi_\phi(x')\} = \delta^3(x-x'). \quad (2)$$

The perturbation of the ADM variables reads

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij} \quad \text{and} \quad \delta \pi^{ij} = \pi^{ij} - \bar{\pi}^{ij}, \quad (3)$$

where the overlined terms represent background quantities. For the lapse and shifts we have

$$N^\mu \rightarrow \bar{N}^\mu + \delta N^\mu. \quad (4)$$

The perturbed Hamiltonian is found to read

$$\mathbb{H} = \bar{N} \mathcal{H}_0^{(0)} + \int_{\mathbb{T}^3} \left(\bar{N} \mathcal{H}_0^{(2)} + \delta N^\mu \delta \mathcal{H}_\mu \right) d^3x, \quad (5)$$

where $\mathcal{H}^{(0)}$, $\delta \mathcal{H}_\mu$, and $\mathcal{H}_0^{(2)}$ are zeroth, first and second order Hamiltonians. The subscript 0 and i distinguish between the scalar and vector part, and it is assumed that Greek indexes run from 0, ..., 3 and Latin ones indicate 1, 2, 3. Moreover, N^μ represent the lapse and shifts for $\mu = 0$ and $\mu = 1, 2, 3$ respectively. The zeroth and first order Hamiltonian are constraints in our theory, whilst the second order Hamiltonian is not, which differs from the non-perturbative approach in which the Hamiltonian is a sum of constraints. It is then straightforward that the lapse and shifts play the role of Lagrange multipliers.

The Hamiltonian (5) defines a gauge system since the dynamics is truncated at linear order.

¹ K^{ij} denotes the extrinsic curvature tensor.

3. Mode decomposition

We will now work in the momentum representation. The Fourier transform of a perturbation variable reads

$$\delta\tilde{X}(\underline{k}) = \int \delta X(\bar{x}) e^{-ik_i x^i} d^3x. \quad (6)$$

From now on, we will raise and lower indexes with the rescaled spatial metric

$$\gamma_{ij} = a^{-2} q_{ij} = a^{-2} a_i^2 \delta_{ij}. \quad (7)$$

Notice that (7) is dynamical in a Bianchi I universe since the directional scale factor a_i is time-dependent. This is not true in the FLRW case in which γ_{ij} is constant. It is easy to see that the time dependence of γ_{ij} is such that the vector $\hat{k}^i$, normal to the wavefront of the plane wave, will also evolve in time. Given $\hat{k}^i$, we can define an orthonormal spatial triad made from the normalized vectors $(\hat{k}^i, \hat{v}^i, \hat{w}^i)$.

Any spatial symmetric 2-rank covariant tensor can be decomposed in scalar, vector, and tensor modes using a γ -orthogonal basis defined as follows:

$$\begin{aligned} A_{ij}^1 &= \gamma_{ij}, & A_{ij}^2 &= \hat{k}_i \hat{k}_j - \frac{1}{3} \gamma_{ij}, \\ A_{ij}^3 &= \frac{1}{\sqrt{2}} (\hat{k}_i \hat{v}_j + \hat{v}_i \hat{k}_j), & A_{ij}^4 &= \frac{1}{\sqrt{2}} (\hat{k}_i \hat{w}_j + \hat{w}_i \hat{k}_j), \\ A_{ij}^5 &= \frac{1}{\sqrt{2}} (\hat{v}_i \hat{w}_j + \hat{w}_i \hat{v}_j), & A_{ij}^6 &= \frac{1}{\sqrt{2}} (\hat{v}_i \hat{v}_j - \hat{w}_i \hat{w}_j). \end{aligned} \quad (8)$$

As expected, in a Bianchi I universe, this basis is time-dependent, while it is constant in FLRW [5].

In the new basis, the ADM variables transform as²

$$\delta q_{ij} = \delta q_n A_{ij}^n \quad \text{and} \quad \delta \pi^{ij} = \delta \pi^n A_n^{ij}. \quad (9)$$

Being this a time-dependent transformation, it adds an extra term in the Hamiltonian whose form can be obtained by computing the symplectic form

$$d\tilde{q}_{ij} \wedge d\tilde{\pi}^{ij} = d\delta q_n \wedge d\delta \pi^n + dt \wedge d \underbrace{\left(\frac{dA_{ij}^n}{dt} A_m^{ij} \delta q_n \delta \pi^m \right)}_{-H_{\text{ext}}}. \quad (10)$$

² The dual basis A_n^{ij} is defined such that $A_n^{ij} A_{ij}^m = \delta_n^m$.

4. Fermi–Walker basis

As mentioned in the previous section, the A basis is constant in FLRW, this means that the triad obtained from the wave-vector k is constant and can be fixed in a convenient position. This, however, is not true in a Bianchi I universe, in which the triad rotates. Using the Fermi–Walker transport to parallel-transport the pair of vectors v and w , we are able to get the simplest dynamical law and the least amount of couplings in the description of our system in this basis. The two vectors v and w are Fermi transported along a future-directed null vector p

$$\vec{p} = \vec{k} + |\vec{k}|\partial_\eta \quad (11)$$

with the metric

$$ds^2 = -d\eta^2 + \gamma_{ij}dx^i dx^j. \quad (12)$$

A Fermi propagated field $\vec{E}$ is given by $\nabla_{\vec{p}}\vec{E} = 0$ and its components read

$$\frac{dE^0}{d\lambda} = -k^i \sigma_{ij} E^j, \quad \frac{dE^j}{d\lambda} = -|\vec{k}| \sigma^j_i E^i, \quad (13)$$

where σ_{ij} is the shear in the spatial direction i, j and λ is an affine parameter. In the case in which the vector field $\vec{E}$ is initially on the plane (v, w) , it will eventually develop longitudinal and temporal components. Thus we need to project the covariant derivative into the plane

$$\frac{dE^j}{d\eta} = -\sigma^j_i E^i + \hat{k}^j \sigma_{ki} E^i, \quad (14)$$

where η is the conformal time and we used the relation $d\eta = |\vec{k}|d\lambda$.

Equation (14) allows us to compute the time derivative of the A basis which is needed to compute the extra Hamiltonian of Eq. (10) coming from the time-dependent transformation as explained in the previous section.

5. Dirac method

In our theory, we have 4 gauge fixing conditions δc_μ and 4 first-class constraints $\delta \mathcal{H}^\mu = 0$. Together they form 8 second-class constraints

$$\delta C_\rho = \{\delta c_1, \delta c_2, \delta c_3, \delta c_4, \delta \mathcal{H}_0, \delta \mathcal{H}_k, \delta \mathcal{H}_v, \delta \mathcal{H}_w\}. \quad (15)$$

As a set of second-class constraints, they form an invertible matrix of the commutation relations

$$\det\{\delta C_\rho, \delta C_\sigma\} \neq 0. \quad (16)$$

We can then introduce the Dirac bracket

$$\{\cdot, \cdot\}_D = \{\cdot, \cdot\} - \{\cdot, \delta C_\rho\} \{\delta C_\rho, \delta C_\sigma\}^{-1} \{\delta C_\sigma, \cdot\}, \quad (17)$$

which are the equivalent of the Poisson bracket in a constrained system. Strongly imposing the second-class constraints on the second-order Hamiltonian, we get the physical Hamiltonian

$$H_{\text{phys}} = \left(N\mathcal{H}_0^{(2)} + \delta N^\mu \delta \mathcal{H}_\mu \right) \Big|_{\delta C_\rho=0} = N\mathcal{H}_0^{(2)} \Big|_{\delta C_\rho=0}. \quad (18)$$

The Hamilton equations in the gauge-fixing surface for any background-independent observable $\mathcal{O}$ are generated by the physical Hamiltonian via the Dirac bracket

$$\dot{\mathcal{O}} = \left\{ \mathcal{O}, N\mathcal{H}_0^{(2)} \Big|_{\delta C_\rho=0} \right\}_D. \quad (19)$$

The obtained physical Hamiltonian is in a form similar to the one of a quasi-harmonic oscillator. By shifting and rescaling our ADM variables, we are able to write our Hamiltonian in the form

$$H_{\text{BI}} = \frac{N}{2a} \left[\delta \tilde{\pi}_\phi^2 + \delta \tilde{\pi}_5^2 + \delta \tilde{\pi}_6^2 + (k^2 + U_\phi) \delta \tilde{\phi}^2 + (k^2 + U_5) \delta \tilde{q}_5^2 + (k^2 + U_6) \delta \tilde{q}_6^2 + C_1 \delta \tilde{q}_5 \delta \tilde{q}_6 + C_2 \delta \tilde{q}_5 \delta \tilde{\phi} + C_3 \delta \tilde{q}_6 \delta \tilde{\phi} \right], \quad (20)$$

where $\delta \tilde{q}_i$ and $\delta \tilde{\pi}^i$ are an extension of the Mukhanov–Sasaki variables³. Notice that in Eq. (20), we still have some coupling terms, as expected these terms vanish in the isotropic limit.

6. Dirac observables

To describe our physical Hamiltonian in a gauge-invariant way, it is convenient to introduce the Dirac observables. They are first-order kinematical phase-space observables which weakly commute with the first-class constraints

$$\forall_{\delta \xi^\rho} \{ \delta D_i, \int \delta \xi^\rho \delta \mathcal{H}_\rho \} \approx 0. \quad (21)$$

They form a complete algebra $\{ \delta D_i, \delta D_j \} \approx \delta D_k$, which can be computed both with the Dirac or Poisson bracket

$$\{ \delta D_i, \delta D_j \}_D \approx \{ \delta D_i, \delta D_j \}. \quad (22)$$

³ The Mukhanov–Sasaki variables are defined as the set of variables which satisfy the equation of motion of the Harmonic oscillator. In our case, we cannot get rid of all the coupling terms, so our variables are only an extension of the traditional ones since they satisfy the equation of motion of a quasi-harmonic oscillator.

From Eq. (21) we notice that they can be defined up to a constraint

$$\{\delta D_i + \alpha^\rho \delta \mathcal{H}_\rho, \delta D_j\} \approx \{\delta D_i, \delta D_j\}. \quad (23)$$

The number of Dirac observables is equal to the number of reduced variables parametrizing any gauge-fixing surface. In our case, the phase space has dimension 14 with 4 gauge fixing conditions and 4 constraints, thus we are left with 6 independent Dirac observables. As pictured in Fig. 1, there is a canonical isomorphism between the physical variables, *i.e.* the extended Mukhanov–Sasaki variables, in any gauge-fixing surface and the Dirac observables

$$\left\{ \delta D_i|_{\delta c_\rho=0}, \delta D_j|_{\delta c_\rho=0} \right\}_D \leftrightarrow \left\{ \delta D_i|_{\delta c'_\rho=0}, \delta D_j|_{\delta c'_\rho=0} \right\}_{D'}, \quad (24)$$

Hence, we can write the physical Hamiltonian as

$$H_{\text{BI}} = \frac{N}{2a} \left[\delta P_1^2 + \delta P_2^2 + \delta P_3^2 + (k^2 + U_\phi) \delta Q_3^2 + (k^2 + U_5) \delta Q_1^2 + (k^2 + U_6) \delta Q_2^2 + C_1 \delta Q_1 \delta Q_2 + C_2 \delta Q_1 \delta Q_3 + C_3 \delta Q_2 \delta Q_3 \right]. \quad (25)$$

where δP^i and δQ_i are Dirac observables. The observables describing gravitational waves are $\delta Q_1 = \frac{1}{\sqrt{2}a} \delta q_5 + \frac{2P_{vw}}{aP_{kk}} (\delta q_1 - \frac{1}{3} \delta q_2)$ and $\delta Q_2 = \frac{1}{\sqrt{2}a} \delta q_6 + \frac{P_{vv} - P_{ww}}{aP_{kk}} (\delta q_1 - \frac{1}{3} \delta q_2)$. As we can see, there can be well-defined gauges in which δQ_1 and δQ_2 contain both tensor and scalar modes if the wavefront has a non-vanishing shear component. Therefore, in an anisotropic universe gravitational waves can no longer be identified as purely transverse and traceless metric perturbations.

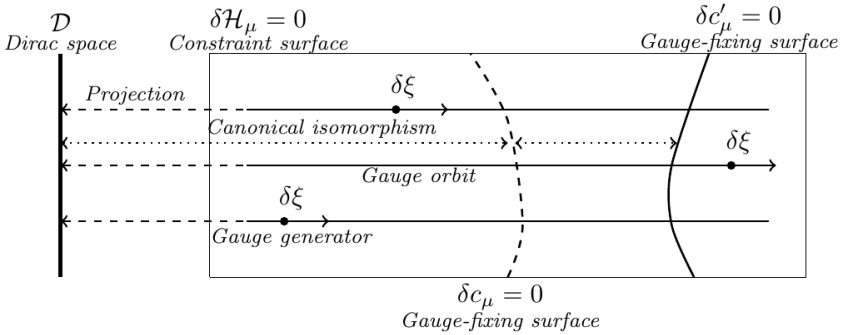


Fig. 1. Basic elements in the Dirac procedure.

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