THERMODYNAMICAL DEFORMATIONS OF INTEGRABLE MODELS OF AdS3 STRINGS*

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Received 28 November 2021, accepted 29 November 2021, published online 31 January 2022

We discuss an integrable model of string on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ in a thermodynamical bath. We show that scattering of the excitations above equilibrium states has some novel features. Thermodynamics points to interesting deformation of the original model for which we discuss finite size effect through mirror TBA equations.

DOI:10.5506/APhysPolBSupp.15.1-A9

1. Intro

The AdS/CFT correspondence [1–3] sparked incredible research activities ranging from the basics of string theory to condensed matter physics [4–7]. It appeared that some string models, including $\mathrm{AdS}_5\times\mathrm{S}_5$, can be realized as an integrable system [8, 9]. Unfortunately, the proposed models have very non-trivial scattering matrices. A reformulation of the theory into the so-called Y-system [10] simplified many calculations considerably. Besides these successes, intricate difficulties discouraged numerous investigators [11].

On the other hand, it was shown that an ordinary bosonic string with one dimension compactified to a circle can be represented as an integrable model with a very simple diagonal scattering matrix [12]. A few years later, sightly modified construction yielded a spectrum of strings on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{T}^4$ [13, 14] and on $\mathrm{AdS}_3\times\mathrm{S}^3\times\mathrm{S}^3\times\mathrm{S}^1$ [15] with the pure NS-NS flux. Since then, the subject has had an interesting development, see *e.g.* recent works [16, 17].

In this paper, we shall dwell upon the models of [13, 14]. Specifically, we generalize this model by terms resulting from thermodynamical considerations. It is known that integrable models have stable excitations above

* Presented at the 7$^{\mathrm{th}}$ Conference of the Polish Society on Relativity, Łódź, Poland, 20–23 September 2021.

thermodynamic equilibrium states but their energy, momenta, and the interaction are subject to finite renormalization called "dressing". Inspired by these results, we introduce a new model modifying chemical potentials and allowing a more general scattering phase. Then we derive formulae for its finite size spectrum.

2. The model

Here, we present the main ingredients of the model under consideration *i.e.* the set of quasi-particles (see the table), their dispersion relations, and the scattering phases defining integrable interactions.

TABLE I

The particle spectrum of pure NS-NS $AdS_3\times S^3\times T^4$ strings integrable model. Different quasi-particles and anti-quasi-particles (entries with bars) will be indexed with capital letters A, B, C and will be called flavour. Quasi-particles $Y, Z, T^{a\dot{a}}, \ldots$ are bosons for which $\psi_A = 0$, while $\eta^a, \chi^a, \ldots$ are fermions ($a, \dot{a} = 1, 2$) for which $\psi_A = i\pi$.

	$Y(p)$	$\bar{Y}(p)$	$Z(p)$	$\bar{Z}(p)$	$T^{a\dot{a}}(p)$	$\eta^a(p)$	$\bar{\eta}^a(p)$	$\chi^{\dot{a}}(p)$	$\bar{\chi}^{\dot{a}}(p)$
μ	1	-1	1	-1	0	1	-1	0	0

Dispersion relation of the modes depends on the flavour A through value of μ

$$H_A^{(0)}(p) = |p + \mu_A| \,. \tag{1}$$

Each flavour A can be chiral $p > -\mu_A$ or anti-chiral $p < -\mu_A$. "Zero-modes" have momenta $p = -\mu_A$ (which are neither chiral nor anti-chiral) and vanishing energy. We shall assume that they will not play any role hereafter.

The scattering phase is diagonal and depends on flavours A, B only through the definition of chirality

$$\Phi_{AB}^{(0)}(p,q) = \mp\frac{2\pi}{k} p\, q\,, \qquad \begin{cases} - & \text{for } p > -\mu_A\,, \quad q < -\mu_B\,, \\ + & \text{for } p < -\mu_A\,, \quad q > -\mu_B\,. \end{cases} \tag{2}$$

We shall also denote with superscript $\pm$ quantities related to the chiral or anti-chiral quasi-particles *e.g.* $p_A^{\pm}$ are momenta $p > -\mu_A$ for $(+)$ and $p < -\mu_A$ for $(-)$, so the scattering phase can be expressed as $\Phi_{AB}^{(0)}(p,q) = -\frac{2\pi}{k}(p_A^+ q_B^- - p_A^- q_B^+)$. The simple product form of (2) leads to dramatic simplifications of the TBA equations which turn to be transcendental equations on some unknown constants.

It is obvious that the scattering phase is not continued at $p = -\mu_A \neq 0$. According to the assumption stated above, we shall ignore terms resulting from differentiation at this discontinuity.

The spectrum of the model at finite but large string size R is given by the Bethe equations (BE) which read

$$p_i R + \sum_{j \neq i} \Phi^{(0)}_{ij}(p_i, p_j) = 2\pi n_i \,. \tag{3}$$

The equations were analysed in [13, 14] and it was shown that their solutions correspond to string spectrum on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathrm{T}^4$ with NS-NS flux.

3. Thermodynamics

Integrable systems have interesting thermodynamical properties [18, 19]. Static behaviour is governed by the TBA equations which yield the density of quasi-particles at equilibrium. One can also find the spectrum of excitations above the equilibrium. They appear to be stable and their scattering matrix is factorizable but their dispersion relation is modified through interaction with the environment. The process in this context is called dressing. Summarizing, we can say that the finite-temperature theory is an integrable deformation of the zero-temperature theory of dressed quasi-particles which interacts with the dressed scattering matrix.

In this section, we shall describe properties of dressed quantities for the model presented in the previous section. Hence, we need to go to thermodynamical limit for BE of the previous section. This leads to the so-called BYE which relates the density of states ρ_{tA} to the density of quasi-particles ρ_A for each flavour A

$$\frac{1}{2\pi} + \sum_B \int \frac{\mathrm{d}q}{2\pi}\, \phi^{(0)}_{AB}(p,q)\rho_B(q) = \rho_{tA}(p)\,, \tag{4}$$

where $\phi^{(0)}_{AB}(p,q) = \partial_p \Phi^{(0)}_{AB}(p,q)$. Equilibrium densities $\rho_B(q)$ are determined by the TBA equations which are integral equations on pseudo-energies $\epsilon_A(q)$. The latter determines equilibrium occupation numbers $n_B(q) = (\mathrm{e}^{\epsilon_B} - \mathrm{e}^{\psi_B})^{-1}$ and densities of quasi-particles

$$\epsilon_A(q) = w_A(q) - \int \frac{\mathrm{d}p}{2\pi} \Lambda_B(p)\; \phi^{(0)}_{BA}(p,q)\,, \tag{5}$$

where $\Lambda_B(p) = -\mathrm{e}^{\psi_B} \log(1 - \mathrm{e}^{\psi_B}\mathrm{e}^{-\epsilon_B(p)})$. In (5) temperature $T = 1/\beta$ appears through

$$w^+_A(q) = \beta\left(q + \mu_A + b^+_A\right)\,, \qquad w^-_A(q) = \beta\left(-(q+\mu_A) + b^-_A\right)\,. \tag{6}$$

In principle, $w(q)$ could be an arbitrary generalized chemical potential as dictated by GGE [20, 21], but here we limit our choice by introducing only the extra "chiral chemical potentials", $b_A^{\pm}$, which will be a free parameter in our considerations. The potentials are necessarily non-zero.

3.1. Dressing

Integrable models have stable excitations above the thermal equilibrium [18, Sec. 8]. The single excitation modes have a dispersion relation given by $\epsilon(p)$ of (5) — these we call dressed energies recalling that the "dressing" is a thermodynamical effect. It appears that not only energy is dressed but also momenta. Thus,

$$\begin{aligned} H_A^{\mathrm{dr}^+}(q) &= \kappa^+(q+\mu_A) + \mathfrak{b}_A^+ , \qquad q > -\mu_A , \\ H_A^{\mathrm{dr}^-}(q) &= -\kappa^-(q+\mu_A) + \mathfrak{b}_A^- , \quad q < -\mu_A . \end{aligned} \tag{7}$$

Due to the simplicity of interaction, the dispersion relation is a renormalized version of (1) with a major impact of potentials (6). The appearance of the couplings $\kappa^{\pm}$ and $\mathfrak{b}_A^{\pm}$ signals possible chiral–anti-chiral disparity.

At large volume, R, dressed quasi-particles scatter with dressed scattering phase Φ^{dr} which differs significantly from the undressed one. It is known [18] that Φ^{dr} respects the equation

$$\Phi_{AB}^{\mathrm{dr}}(p,q) + \sum_C \int \frac{\mathrm{d}r}{2\pi} \partial_r \Phi_{AC}^{(0)}(p,r) n_C(r) \Phi_{CB}^{\mathrm{dr}}(r,q) = \Phi_{AB}^{(0)}(p,q) . \tag{8}$$

The equation can be easily solved by

$$\Phi_{AB}^{\mathrm{dr}}(p,q) = C_{++} p_A^+ q_B^+ + C_{+-} p_A^+ q_B^- + C_{-+} p_A^- q_B^+ + C_{--} p_A^- q_B^- \tag{9}$$

with some non-zero $C_{\pm\pm}$. It is interesting that Φ^{dr} involves interaction between quasi-particles of the same chirality: the effect results from interaction through a thermal bath. In consequence, the corresponding BE is more involved and non-trivial even for single quasi-particle excitation above the bath.

4. Finite size model

The mirror TBA [23, 24] allows to calculate energies of states at finite volume R [25, 26]. The mirror transformation is: $p \to i\hat{H}$, $H \to i\hat{p}$. The mirror scattering phase [23] follows from the dressed one (9) and respects $\hat{\Phi}(\hat{p},\hat{q}) = \Phi^{\mathrm{dr}}(p,q)$. In order to find the energies of states, we need to write TBA for mirror pseudo-energy

$$\hat{\epsilon}_A(\hat{p}) = \psi_A + R\,\hat{H}_A(\hat{p}) - \left[\Lambda_B * \hat{\phi}_{BA}\right] - i\sum_j \hat{\Phi}_{AB_j}(\hat{p},\hat{p}_j)\,, \tag{10}$$

$$\hat{\epsilon}_{A_j}(\hat{p}_j) = -2\pi i\, n_j + \psi_{A_j}\,, \tag{11}$$

where $\hat{\phi}_{BA}(\hat{p},\hat{q}) = \partial_{\hat{p}}\hat{\Phi}_{BA}(\hat{p},\hat{q})$, and

$$\begin{aligned}\left[\Lambda_B * \hat{\phi}_{BA}\right] &= -\left(C_{++}\hat{H}^+_A(\hat{p}) + C_{+-}\hat{H}^-_A(\hat{p})\right)\hat{I}^+ \\ &\quad + \left(C_{-+}\hat{H}^+_A(\hat{p}) + C_{--}\hat{H}^-_A(\hat{p})\right)\hat{I}^-\,.\end{aligned}$$

From (10), we infer that mirror pseudo-energies depend on flavour only through chirality and have the form of

$$\hat{\epsilon}^{\pm}_A(\hat{p}) = \psi_A + g^{\pm}\,\hat{H}^{\pm}_A\left(\hat{p}^{\pm}\right) \tag{12}$$

with two unknown constants $g^{\pm}$ determined by the mirror TBA (10). The constants have an interpretation of effective string size as we shall see soon.

We have introduced two integrals

$$\hat{I}^{\pm} = \frac{1}{\kappa^{\pm}}\sum_A \int_{\hat{c}^{\pm}_A} \frac{\mathrm{d}p}{2\pi}\Lambda_A\left(\hat{\epsilon}^{\pm}_A(p)\right)\,, \tag{13}$$

where $\hat{c}^{\pm}_A$ are: $\hat{c}^{+}_A = \{\mathfrak{b}^{+}_A, +\infty\}$, $\hat{c}^{-}_A = \{-\infty, -\mathfrak{b}^{-}_A\}$. Notice that contrary to the standard kinematics TBA, due to ψ_A appearing in solution (12), bosonic and fermionic quasi-particles contributions differ only by the overall sign. All chiral chemical potentials enter through $\hat{I}^{\pm}$s only. Thus, we can treat $\hat{I}^{\pm}$s as free parameters. For the reality of gs, we need the reality of $\hat{I}^{\pm}$'s. Substituting (12) into the mirror TBA, we get equations on $g^{\pm}$

$$\begin{aligned} g^+ &= R + \left(C_{++}\hat{I}^+ - C_{-+}\hat{I}^-\right) + \left(C_{++}P^+ + C_{+-}P^-\right)\,, \\ g^- &= R + \left(C_{+-}\hat{I}^+ - C_{--}\hat{I}^-\right) + \left(C_{--}P^- + C_{-+}P^+\right)\,. \end{aligned} \tag{14}$$

The above leads to the Bethe equations for momenta of the physical states what is the main result of this work. For supersymmetric backgrounds, $\hat{I}^{\pm} = 0$ for any $g^{\pm}$, thus (14) are just the BE.

Notice that (14) are sums of three terms: the bare size of the string R, the contribution of virtual particles circling around the string, and the total momenta of a given state P^+, P^- *i.e.* sum of moments of all quasi-particles. The second term vanishes for $R \to \infty$ (if $\mathfrak{b}^{\pm}_A/\kappa^{\pm} > 0$).

It is worth to check if the models discussed here correspond to any deformation of the $\mathrm{AdS}_3{\times}\mathrm{S}^3{\times}\mathrm{T}^4$ strings models. We leave this problem to a future publication.

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