

RELATIVISTIC FLUID DYNAMICS OF MULTIPLE CONSERVED CHARGES*

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The equations of multicomponent relativistic second-order dissipative fluid dynamics from the Boltzmann equations for a reactive mixture of N_{spec} particle species with N_q intrinsic quantum numbers such as electric charge, baryon number, and strangeness are presented. We discuss the “single-fluid” description of a multicomponent fluid, which consists of $4 + N_q$ conservation laws closed by $6 + 3N_q$ equations of motion for the dissipative quantities in the $(10 + 4N_q)$ -moment approximation.

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1. Introduction

Fluid-dynamical modeling of relativistic nuclear matter is commonly based on the relativistic second-order dissipative fluid-dynamical theory of Israel and Stewart [1] formulated for a simple fluid, *i.e.* a single-component fluid. However, the matter created in high-energy nuclear collisions is fundamentally of multicomponent nature consisting of different types of particle species carrying multiple conserved quantum numbers such as baryon number B , electric charge Q , and strangeness S . Further, the generated charge

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currents are directly coupled to each other. Therefore, a relativistic fluid-dynamical theory that accounts for multiple conserved charges in heavy-ion collisions is necessary. We derive the continuity equations for each conserved quantum charge as well as the conservation laws for total energy and momentum in the single-fluid approximation. These equations are closed by providing second-order equations of motion in the $(10 + 4N_q)$ -moment approximation for the dissipative quantities [2].

2. The Boltzmann equation for a reactive mixture

In kinetic theory, a mixture of N_{spec} different particle species or chemical components is characterized by the single-particle distribution functions for each particle species i , labeled by a lower index, $f(x, k_i) \equiv f_{i,\mathbf{k}}$. The space-time evolution of the distribution function of species i is given by the relativistic Boltzmann equation [3, 4]

$$k_i^\mu \partial_\mu f_{i,\mathbf{k}} \equiv C_i(x, k_i) = \sum_{j=1}^{N_{\text{spec}}} C_{ij}[f], \quad (1)$$

where the four-momentum of species i is denoted by k_i^μ , and it is normalized to the corresponding rest mass squared, $k_i^\mu k_{i,\mu} = m_i^2$. In the case of binary inelastic, *i.e.* reactive collisions, the initial and final particles species may be different, $i + j \rightarrow a + b$, such that the collision term reads

$$C_{ij}[f] = \frac{1}{2} \sum_{a,b=1}^{N_{\text{spec}}} \int dK'_j dP'_a dP'_b W_{ab \rightarrow ij}^{pp' \rightarrow kk'} \left(f_{a,\mathbf{p}} f_{b,\mathbf{p}'} \tilde{f}_{i,\mathbf{k}} \tilde{f}_{j,\mathbf{k}'} - f_{i,\mathbf{k}} f_{j,\mathbf{k}'} \tilde{f}_{a,\mathbf{p}} \tilde{f}_{b,\mathbf{p}'} \right), \quad (2)$$

where $W_{ij \rightarrow ab}^{kk' \rightarrow pp'}$ are the transition probabilities, while $\tilde{f}_{i,\mathbf{k}} = 1 - a_i f_{i,\mathbf{k}}/g_i$, with $a_i = \pm 1$ for fermions/bosons, and $a_i \rightarrow 0$ for classical particles, respectively. Here, g_i is the spin degeneracy of particle species i , while the Lorentz-invariant integration measure is $dK_i = d^3\mathbf{k}_i/[(2\pi)^3 k_i^0]$.

The single-particle distribution function $f_{i,\mathbf{k}}$ can be decomposed into an equilibrium part, $f_{i,\mathbf{k}}^{(0)}$, and an out-of-equilibrium part, $\delta f_{i,\mathbf{k}}$, as

$$f_{i,\mathbf{k}} \equiv f_{i,\mathbf{k}}^{(0)} + \delta f_{i,\mathbf{k}} = g_i [\exp(\beta E_{i,\mathbf{k}} - \alpha_i) + a_i]^{-1} + \delta f_{i,\mathbf{k}}, \quad (3)$$

where the (Lorentz-invariant) energy of a particle of species i is defined as $E_{i,\mathbf{k}} = k_i^\mu u_\mu$, while u^μ is the fluid flow velocity normalized to unity $u^\mu u_\mu = 1$. Above, the local-equilibrium distribution function of species i is given by the Jüttner distribution function [3, 4], where $T \equiv 1/\beta$ is the temperature and $\mu_i \equiv \beta\alpha_i$ is the chemical potential of species i .

In various inelastic collisions, the particle number corresponding to a given species is not conserved due to particle creation and annihilation processes or various chemical reactions. Therefore, only a few intrinsic quantum numbers, such as electric charge, baryon number, and strangeness are conserved. Hence, the chemical potential μ_i of a given particle i may be expressed in terms of N_q chemical potentials of conserved quantum “charges”

$$\mu_i(\{\mu_q\}) \equiv \sum_q^{\{B,Q,S\}} q_i \mu_q = B_i \mu_B + Q_i \mu_Q + S_i \mu_S, \quad (4)$$

where $\{\mu_q\} \equiv \{\mu_B, \mu_Q, \mu_S\}$, with μ_B , μ_Q , and μ_S , being the baryon, electric, and strangeness chemical potentials, respectively, while B_i , Q_i , and S_i are the baryon number, electric charge, and strangeness of species i . Furthermore, $q_i \equiv \partial \mu_i(\{\mu_{q'}\}) / \partial \mu_q$ denotes the intrinsic quantum number of species i .

The tensor decompositions with respect to the time-like normalized flow velocity u^μ being the eigenvector of $T^{\mu\nu} u_\nu = e u^\mu$, and summed over all particle species, lead to the total fluid-dynamical quantities of the mixture

$$N_q^\mu \equiv \sum_{i=1}^{N_{\text{spec}}} q_i N_i^\mu = \sum_{i=1}^{N_{\text{spec}}} [q_i n_i u^\mu + q_i V_i^\mu] \equiv n_q u^\mu + V_q^\mu, \quad (5)$$

$$\begin{aligned} T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\text{spec}}} T_i^{\mu\nu} = \sum_{i=1}^{N_{\text{spec}}} [e_i u^\mu u^\nu - (P_i + \Pi_i) \Delta^{\mu\nu} + \pi_i^{\mu\nu}] \\ &\equiv e u^\mu u^\nu - (P + \Pi) \Delta^{\mu\nu} + \pi^{\mu\nu}. \end{aligned} \quad (6)$$

The net-charge density, the energy density, and the pressure in equilibrium as well as the bulk viscous pressure of the mixture are

$$n_q \equiv N_q^\mu u_\mu = \sum_{i=1}^{N_{\text{spec}}} q_i \langle E_{\mathbf{k}} \rangle_{i,0}, \quad e \equiv T^{\mu\nu} u_\mu u_\nu = \sum_{i=1}^{N_{\text{spec}}} \langle E_{\mathbf{k}}^2 \rangle_{i,0}, \quad (7)$$

$$P + \Pi \equiv -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} \sum_{i=1}^{N_{\text{spec}}} \left[\langle \Delta_{\mu\nu} k^\mu k^\nu \rangle_{i,0} + \langle \Delta_{\mu\nu} k^\mu k^\nu \rangle_{i,\delta} \right], \quad (8)$$

where $\langle \cdots \rangle_{i,0} \equiv \int dK_i (\cdots)_i f_{i,\mathbf{k}}^{(0)}$ and $\langle \cdots \rangle_{i,\delta} \equiv \int dK_i (\cdots)_i \delta f_{i,\mathbf{k}}$ denotes equilibrium and out-of-equilibrium momentum integrals. The thermodynamic variables in an arbitrary state not too far from the local equilibrium are the same as the net-charge densities and the total energy density in some fictitious local-equilibrium reference state. An equation of state determines these thermodynamic quantities as functions of temperature and chemical potentials, *i.e.* $n_q = n_q(T, \{\mu_q\})$, $e = e(T, \{\mu_q\})$, and $P = P(T, \{\mu_q\})$.

The net-particle diffusion, the net-charge diffusion, and the shear-stress tensor of the mixture are

$$V_q^\mu \equiv \Delta_\nu^\mu N_q^\nu = \sum_{i=1}^{N_{\text{spec}}} q_i \left\langle k^{(\mu)} \right\rangle_{i,\delta}, \quad \pi^{\mu\nu} \equiv \Delta_{\alpha\beta}^{\mu\nu} T^{\alpha\beta} = \sum_{i=1}^{N_{\text{spec}}} \left\langle k^{(\mu} k^{\nu)} \right\rangle_{i,\delta}, \quad (9)$$

where $k^{(\mu)} = \Delta_\nu^\mu k^\nu$ is the orthogonal projection, while $k^{(\mu} k^{\nu)} = \Delta_{\alpha\beta}^{\mu\nu} k^\alpha k^\beta$ is the symmetric, orthogonal and traceless projection, respectively.

The fluid-dynamical fields of the mixture are a combination of multiple particle species where the number of particles of an individual species may or may not be conserved. The mixture will be treated as a single fluid such that its space-time evolution is governed by a single velocity field u^μ , and thus, it is determined in terms of the total energy-momentum tensor $T^{\mu\nu}$ and the charge four-currents N_q^μ , with $10 + 4N_q$ fluid-dynamical fields.

In binary collisions, the net charges as well as the energy and momentum of particles are conserved, and the equations of fluid dynamics of a mixture may be derived from the Boltzmann equation (1). The N_q charge-conservation equations read

$$\begin{aligned} \partial_\mu N_q^\mu &\equiv \sum_{i=1}^{N_{\text{spec}}} q_i Dn_i + \sum_{i=1}^{N_{\text{spec}}} q_i n_i \theta + \sum_{i=1}^{N_{\text{spec}}} q_i \partial_\mu V_i^\mu \\ &= Dn_q + n_q \theta + \partial_\mu V_q^\mu = 0, \end{aligned} \quad (10)$$

where $D = u^\mu \partial_\mu$ or an overdot denotes the proper-time derivative and $\theta = \partial_\mu u^\mu$ is the expansion scalar. The conservation equations of total energy and total momentum of the mixture are

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\text{spec}}} D e_i + \sum_{i=1}^{N_{\text{spec}}} (e_i + P_i + \Pi_i) \theta - \sum_{i=1}^{N_{\text{spec}}} \pi_i^{\mu\nu} \sigma_{\mu\nu} \\ &= D e + (e + P + \Pi) \theta - \pi^{\mu\nu} \sigma_{\mu\nu} = 0, \end{aligned} \quad (11)$$

$$\begin{aligned} \Delta_\beta^\mu \partial_\alpha T^{\alpha\beta} &\equiv \sum_{i=1}^{N_{\text{spec}}} (e_i + P_i + \Pi_i) D u^\mu - \nabla^\mu \sum_{i=1}^{N_{\text{spec}}} (P_i + \Pi_i) + \Delta_\beta^\mu \partial_\alpha \sum_{i=1}^{N_{\text{spec}}} \pi_i^{\alpha\beta} \\ &= (e + P + \Pi) D u^\mu - \nabla^\mu (P + \Pi) + \Delta_\beta^\mu \partial_\alpha \pi^{\alpha\beta} = 0, \end{aligned} \quad (12)$$

where $\sigma^{\mu\nu} = \nabla^{(\mu} u^{\nu)}$ is the shear-stress tensor. Therefore, in a dissipative mixture of N_q conserved charges, we have only $4 + N_q$ conservation equations. The additional $6 + 3N_q$ equations for the dissipative fields Π , V_q^μ , and $\pi^{\mu\nu}$ are derived from the Boltzmann equation, see Ref. [2] for more details.

The resulting equation of motion for the bulk viscous pressure is

$$\begin{aligned} \tau_{II} \dot{\Pi} + \Pi = & -\zeta\theta - \delta_{II\Pi} \Pi\theta + \lambda_{II\pi} \pi^{\mu\nu} \sigma_{\mu\nu} - \sum_q^{\{B,Q,S\}} \ell_{IIV}^{(q)} \nabla_\mu V_q^\mu \\ & - \sum_q^{\{B,Q,S\}} \tau_{IIV}^{(q)} V_q^\mu \dot{u}_\mu - \sum_{q,q'}^{\{B,Q,S\}} \lambda_{IIV}^{(q,q')} V_q^\mu \nabla_\mu \alpha_{q'}. \end{aligned} \quad (13)$$

Similarly, the equations of motion for the charge diffusion currents read

$$\begin{aligned} \sum_q^{\{B,Q,S\}} \tau_{q'q} \dot{V}_q^{\langle\mu\rangle} + V_{q'}^\mu = & \sum_q^{\{B,Q,S\}} \kappa_{q'q} \nabla^\mu \alpha_q - \sum_q^{\{B,Q,S\}} \tau_{q'q} V_{q,\nu} \omega^{\nu\mu} \\ & - \sum_q^{\{B,Q,S\}} \delta_{VV}^{(q',q)} V_q^\mu \theta - \sum_q^{\{B,Q,S\}} \lambda_{VV}^{(q',q)} V_{q,\nu} \sigma^{\mu\nu} \\ & - \ell_{V\Pi}^{(q')} \nabla^\mu \Pi + \ell_{V\pi}^{(q')} \Delta^{\mu\nu} \nabla_\lambda \pi_\nu^\lambda + \tau_{V\Pi}^{(q')} \Pi \dot{u}^\mu - \tau_{V\pi}^{(q')} \pi^{\mu\nu} \dot{u}_\nu \\ & + \sum_q^{\{B,Q,S\}} \lambda_{V\Pi}^{(q',q)} \Pi \nabla^\mu \alpha_q - \sum_q^{\{B,Q,S\}} \lambda_{V\pi}^{(q',q)} \pi^{\mu\nu} \nabla_\nu \alpha_q, \end{aligned} \quad (14)$$

while the equation of motion for the shear-stress tensor leads

$$\begin{aligned} \tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = & 2\eta\sigma^{\mu\nu} + 2\tau_\pi \pi_\lambda^{\langle\mu} \omega^{\nu\rangle\lambda} - \delta_{\pi\pi} \pi^{\mu\nu} \theta - \tau_{\pi\pi} \pi^{\lambda\langle\mu} \sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi} \Pi \sigma^{\mu\nu} \\ & - \sum_q^{\{B,Q,S\}} \tau_{\pi V}^{(q)} V_q^{\langle\mu} \dot{u}^{\nu\rangle} + \sum_q^{\{B,Q,S\}} \ell_{\pi V}^{(q)} \nabla^{\langle\mu} V_q^{\nu\rangle} \\ & + \sum_{q,q'}^{\{B,Q,S\}} \lambda_{\pi V}^{(q,q')} V_q^{\langle\mu} \nabla^{\nu\rangle} \alpha_{q'}. \end{aligned} \quad (15)$$

Equations of motion (13), (14), and (15) are of relaxation-type similarly to the second-order theories of Israel and Stewart [1], and are identical to those found earlier in Refs. [5, 6]. Furthermore, these equations are formally similar to the relaxation equations of a single-component system but feature different transport coefficients, which contain the microscopic interactions of all components. For a single-component fluid, *i.e.* for $N_{\text{spec}} = N_q = 1$, the results of Ref. [7] are identically reproduced.

Here, the first-order transport coefficients of the mixture are: the bulk viscosity ζ , the diffusion coefficients $\kappa_{qq'}$, and the shear viscosity η . These are obtained by summing over all particle species

$$\zeta \equiv - \sum_{s=1}^{N_{\text{spec}}} \frac{m_s^2}{3} \zeta_{s,0}, \quad \eta \equiv \sum_{s=1}^{N_{\text{spec}}} \eta_{s,0}, \quad \kappa_{qq'} \equiv \sum_{s=1}^{N_{\text{spec}}} q_s \kappa_{s,0,q'}. \quad (16)$$

The relaxation time of the bulk viscos pressure τ_{Π} , the diffusion currents, $\tau_{qq'}$, and the stress tensor τ_{π} are related to the inverse of the linearized collision term (2), while all the remaining second-order transport coefficients are listed in Ref. [2].

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