RELATIVISTIC FLUID DYNAMICS OF MULTIPLE CONSERVED CHARGES*

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The equations of multicomponent relativistic second-order dissipative fluid dynamics from the Boltzmann equations for a reactive mixture of $N_{\rm spec}$ particle species with N_q intrinsic quantum numbers such as electric charge, baryon number, and strangeness are presented. We discuss the "single-fluid" description of a multicomponent fluid, which consists of $4+N_q$ conservation laws closed by $6+3N_q$ equations of motion for the dissipative quantities in the $(10+4N_q)$-moment approximation.

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1. Introduction

Fluid-dynamical modeling of relativistic nuclear matter is commonly based on the relativistic second-order dissipative fluid-dynamical theory of Israel and Stewart [1] formulated for a simple fluid, *i.e.* a single-component fluid. However, the matter created in high-energy nuclear collisions is fundamentally of multicomponent nature consisting of different types of particle species carrying multiple conserved quantum numbers such as baryon number B, electric charge Q, and strangeness S. Further, the generated charge

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currents are directly coupled to each other. Therefore, a relativistic fluid-dynamical theory that accounts for multiple conserved charges in heavy-ion collisions is necessary. We derive the continuity equations for each conserved quantum charge as well as the conservation laws for total energy and momentum in the single-fluid approximation. These equations are closed by providing second-order equations of motion in the $(10+4N_q)$-moment approximation for the dissipative quantities [2].

2. The Boltzmann equation for a reactive mixture

In kinetic theory, a mixture of $N_{\rm spec}$ different particle species or chemical components is characterized by the single-particle distribution functions for each particle species i, labeled by a lower index, $f(x,k_i) \equiv f_{i,\boldsymbol{k}}$. The space-time evolution of the distribution function of species i is given by the relativistic Boltzmann equation [3, 4]

$$k_i^\mu \partial_\mu f_{i,\boldsymbol{k}} \equiv C_i(x,k_i) = \sum_{j=1}^{N_{\rm spec}} C_{ij}[f] \,, \tag{1}$$

where the four-momentum of species i is denoted by k_i^μ, and it is normalized to the corresponding rest mass squared, $k_i^\mu k_{i,\mu} = m_i^2$. In the case of binary inelastic, *i.e.* reactive collisions, the initial and final particles species may be different, $i+j \to a+b$, such that the collision term reads

$$C_{ij}[f] = \frac{1}{2}\sum_{a,b=1}^{N_{\rm spec}} \int {\rm d}K_j' {\rm d}P_a {\rm d}P_b' W_{ab\to ij}^{pp'\to kk'} \Big(f_{a,\boldsymbol{p}} f_{b,\boldsymbol{p}'} \tilde{f}_{i,\boldsymbol{k}} \tilde{f}_{j,\boldsymbol{k}'} - f_{i,\boldsymbol{k}} f_{j,\boldsymbol{k}'} \tilde{f}_{a,\boldsymbol{p}} \tilde{f}_{b,\boldsymbol{p}'} \Big) \,, \tag{2}$$

where $W_{ij\to ab}^{kk'\to pp'}$ are the transition probabilities, while $\tilde{f}_{i,\boldsymbol{k}} = 1 - a_i f_{i,\boldsymbol{k}}/g_i$, with $a_i = \pm 1$ for fermions/bosons, and $a_i \to 0$ for classical particles, respectively. Here, g_i is the spin degeneracy of particle species i, while the Lorentz-invariant integration measure is ${\rm d}K_i = {\rm d}^3\boldsymbol{k}_i/[(2\pi)^3 k_i^0]$.

The single-particle distribution function $f_{i,\boldsymbol{k}}$ can be decomposed into an equilibrium part, $f_{i,\boldsymbol{k}}^{(0)}$, and an out-of-equilibrium part, $\delta f_{i,\boldsymbol{k}}$, as

$$f_{i,\boldsymbol{k}} \equiv f_{i,\boldsymbol{k}}^{(0)} + \delta f_{i,\boldsymbol{k}} = g_i \left[\exp\left(\beta E_{i,\boldsymbol{k}} - \alpha_i\right) + a_i\right]^{-1} + \delta f_{i,\boldsymbol{k}} \,, \tag{3}$$

where the (Lorentz-invariant) energy of a particle of species i is defined as $E_{i,\boldsymbol{k}} = k_i^\mu u_\mu$, while u^μ is the fluid flow velocity normalized to unity $u^\mu u_\mu = 1$. Above, the local-equilibrium distribution function of species i is given by the Jüttner distribution function [3, 4], where $T \equiv 1/\beta$ is the temperature and $\mu_i \equiv \beta\alpha_i$ is the chemical potential of species i.

In various inelastic collisions, the particle number corresponding to a given species is not conserved due to particle creation and annihilation processes or various chemical reactions. Therefore, only a few intrinsic quantum numbers, such as electric charge, baryon number, and strangeness are conserved. Hence, the chemical potential μ_i of a given particle i may be expressed in terms of N_q chemical potentials of conserved quantum "charges"

$$\mu_i\left(\{\mu_q\}\right) \equiv \sum_q^{\{B,Q,S\}} q_i \mu_q = B_i \mu_B + Q_i \mu_Q + S_i \mu_S \,, \tag{4}$$

where $\{\mu_q\} \equiv \{\mu_B, \mu_Q, \mu_S\}$, with μ_B, μ_Q, and μ_S, being the baryon, electric, and strangeness chemical potentials, respectively, while B_i, Q_i, and S_i are the baryon number, electric charge, and strangeness of species. Furthermore, $q_i \equiv \partial \mu_i(\{\mu_{q'}\})/\partial \mu_q$ denotes the intrinsic quantum number of species i.

The tensor decompositions with respect to the time-like normalized flow velocity u^μ being the eigenvector of $T^{\mu\nu}u_\nu = eu^\mu$, and summed over all particle species, lead to the total fluid-dynamical quantities of the mixture

$$N_q^\mu \equiv \sum_{i=1}^{N_{\rm spec}} q_i N_i^\mu = \sum_{i=1}^{N_{\rm spec}} \left[q_i n_i u^\mu + q_i V_i^\mu\right] \equiv n_q u^\mu + V_q^\mu \,, \tag{5}$$

$$\begin{aligned} T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\rm spec}} T_i^{\mu\nu} = \sum_{i=1}^{N_{\rm spec}} \left[e_i u^\mu u^\nu - \left(P_i + \Pi_i\right) \Delta^{\mu\nu} + \pi_i^{\mu\nu}\right] \\ &\equiv eu^\mu u^\nu - \left(P + \Pi\right) \Delta^{\mu\nu} + \pi^{\mu\nu} \,. \end{aligned} \tag{6}$$

The net-charge density, the energy density, and the pressure in equilibrium as well as the bulk viscous pressure of the mixture are

$$n_q \equiv N_q^\mu u_\mu = \sum_{i=1}^{N_{\rm spec}} q_i \left\langle E_{\boldsymbol{k}}\right\rangle_{i,0} \,, \qquad e \equiv T^{\mu\nu} u_\mu u_\nu = \sum_{i=1}^{N_{\rm spec}} \left\langle E_{\boldsymbol{k}}^2\right\rangle_{i,0} \,, \tag{7}$$

$$P + \Pi \equiv -\frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu} = -\frac{1}{3} \sum_{i=1}^{N_{\rm spec}} \left[\left\langle \Delta_{\mu\nu} k^\mu k^\nu\right\rangle_{i,0} + \left\langle \Delta_{\mu\nu} k^\mu k^\nu\right\rangle_{i,\delta}\right] , \tag{8}$$

where $\langle \cdots \rangle_{i,0} \equiv \int {\rm d}K_i \left(\cdots\right)_i f_{i,\boldsymbol{k}}^{(0)}$ and $\langle \cdots \rangle_{i,\delta} \equiv \int {\rm d}K_i \left(\cdots\right)_i \delta f_{i,\boldsymbol{k}}$ denotes equilibrium and out-of-equilibrium momentum integrals. The thermodynamic variables in an arbitrary state not too far from the local equilibrium are the same as the net-charge densities and the total energy density in some fictitious local-equilibrium reference state. An equation of state determines these thermodynamic quantities as functions of temperature and chemical potentials, *i.e.* $n_q = n_q\left(T, \{\mu_q\}\right)$, $e = e\left(T, \{\mu_q\}\right)$, and $P = P\left(T, \{\mu_q\}\right)$.

The net-particle diffusion, the net-charge diffusion, and the shear-stress tensor of the mixture are

$$V_q^\mu \equiv \Delta^\mu_\nu N_q^\nu = \sum_{i=1}^{N_{\rm spec}} q_i \left\langle k^{\langle\mu\rangle}\right\rangle_{i,\delta} , \qquad \pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta} = \sum_{i=1}^{N_{\rm spec}} \left\langle k^{\langle\mu} k^{\nu\rangle}\right\rangle_{i,\delta} , \tag{9}$$

where $k^{\langle\mu\rangle} = \Delta^\mu_\nu k^\nu$ is the orthogonal projection, while $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$ is the symmetric, orthogonal and traceless projection, respectively.

The fluid-dynamical fields of the mixture are a combination of multiple particle species where the number of particles of an individual species may or may not be conserved. The mixture will be treated as a single fluid such that its space-time evolution is governed by a single velocity field u^μ, and thus, it is determined in terms of the total energy-momentum tensor $T^{\mu\nu}$ and the charge four-currents N_q^μ, with $10 + 4N_q$ fluid-dynamical fields.

In binary collisions, the net charges as well as the energy and momentum of particles are conserved, and the equations of fluid dynamics of a mixture may be derived from the Boltzmann equation (1). The N_q charge-conservation equations read

$$\begin{aligned} \partial_\mu N_q^\mu &\equiv \sum_{i=1}^{N_{\rm spec}} q_i D n_i + \sum_{i=1}^{N_{\rm spec}} q_i n_i \theta + \sum_{i=1}^{N_{\rm spec}} q_i \partial_\mu V_i^\mu \\ &= D n_q + n_q \theta + \partial_\mu V_q^\mu = 0 \,, \end{aligned} \tag{10}$$

where $D = u^\mu \partial_\mu$ or an overdot denotes the proper-time derivative and $\theta = \partial_\mu u^\mu$ is the expansion scalar. The conservation equations of total energy and total momentum of the mixture are

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &\equiv \sum_{i=1}^{N_{\rm spec}} D e_i + \sum_{i=1}^{N_{\rm spec}} \left(e_i + P_i + \Pi_i\right) \theta - \sum_{i=1}^{N_{\rm spec}} \pi_i^{\mu\nu} \sigma_{\mu\nu} \\ &= D e + \left(e + P + \Pi\right) \theta - \pi^{\mu\nu} \sigma_{\mu\nu} = 0 \,, \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta^\mu_\beta \partial_\alpha T^{\alpha\beta} &\equiv \sum_{i=1}^{N_{\rm spec}} \left(e_i + P_i + \Pi_i\right) D u^\mu - \nabla^\mu \sum_{i=1}^{N_{\rm spec}} \left(P_i + \Pi_i\right) + \Delta^\mu_\beta \partial_\alpha \sum_{i=1}^{N_{\rm spec}} \pi_i^{\alpha\beta} \\ &= \left(e + P + \Pi\right) D u^\mu - \nabla^\mu \left(P + \Pi\right) + \Delta^\mu_\beta \partial_\alpha \pi^{\alpha\beta} = 0 \,, \end{aligned} \tag{12}$$

where $\sigma^{\mu\nu} = \nabla^{\langle\mu} u^{\nu\rangle}$ is the shear-stress tensor. Therefore, in a dissipative mixture of N_q conserved charges, we have only $4+N_q$ conservation equations. The additional $6 + 3N_q$ equations for the dissipative fields Π, V_q^μ, and $\pi^{\mu\nu}$ are derived from the Boltzmann equation, see Ref. [2] for more details.

The resulting equation of motion for the bulk viscous pressure is

$$
\begin{aligned}
\tau_\Pi \dot{\Pi} + \Pi \;&=\; -\zeta\theta - \delta_{\Pi\Pi}\,\Pi\theta + \lambda_{\Pi\pi}\,\pi^{\mu\nu}\sigma_{\mu\nu} - \sum_q^{\{B,Q,S\}} \ell_{\Pi V}^{(q)}\,\nabla_\mu V_q^\mu \\
&\quad - \sum_q^{\{B,Q,S\}} \tau_{\Pi V}^{(q)}\,V_q^\mu \dot{u}_\mu - \sum_{q,q'}^{\{B,Q,S\}} \lambda_{\Pi V}^{(q,q')}\,V_q^\mu \nabla_\mu \alpha_{q'}\,. \qquad (13)
\end{aligned}
$$

Similarly, the equations of motion for the charge diffusion currents read

$$
\begin{aligned}
\sum_q^{\{B,Q,S\}} \tau_{q'q}\,\dot{V}_q^{\langle\mu\rangle} + V_{q'}^\mu &= \sum_q^{\{B,Q,S\}} \kappa_{q'q}\,\nabla^\mu \alpha_q - \sum_q^{\{B,Q,S\}} \tau_{q'q}\,V_{q,\nu}\omega^{\nu\mu} \\
&\quad - \sum_q^{\{B,Q,S\}} \delta_{VV}^{(q',q)}\,V_q^\mu\theta - \sum_q^{\{B,Q,S\}} \lambda_{VV}^{(q',q)}\,V_{q,\nu}\sigma^{\mu\nu} \\
&\quad -\ell_{V\Pi}^{(q')}\,\nabla^\mu \Pi + \ell_{V\pi}^{(q')}\,\Delta^{\mu\nu}\nabla_\lambda \pi_\nu^\lambda + \tau_{V\Pi}^{(q')}\,\Pi\dot{u}^\mu - \tau_{V\pi}^{(q')}\,\pi^{\mu\nu}\dot{u}_\nu \\
&\quad + \sum_q^{\{B,Q,S\}} \lambda_{V\Pi}^{(q',q)}\,\Pi\nabla^\mu\alpha_q - \sum_q^{\{B,Q,S\}} \lambda_{V\pi}^{(q',q)}\,\pi^{\mu\nu}\nabla_\nu\alpha_q\,, \qquad (14)
\end{aligned}
$$

while the equation of motion for the shear-stress tensor leads

$$
\begin{aligned}
\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} &= 2\eta\sigma^{\mu\nu} + 2\tau_\pi\,\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\,\pi^{\mu\nu}\theta - \tau_{\pi\pi}\,\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\,\Pi\sigma^{\mu\nu} \\
&\quad - \sum_q^{\{B,Q,S\}} \tau_{\pi V}^{(q)}\,V_q^{\langle\mu}\dot{u}^{\nu\rangle} + \sum_q^{\{B,Q,S\}} \ell_{\pi V}^{(q)}\,\nabla^{\langle\mu}V_q^{\nu\rangle} \\
&\quad + \sum_{q,q'}^{\{B,Q,S\}} \lambda_{\pi V}^{(q,q')}\,V_q^{\langle\mu}\nabla^{\nu\rangle}\alpha_{q'}\,. \qquad (15)
\end{aligned}
$$

Equations of motion (13), (14), and (15) are of relaxation-type similarly to the second-order theories of Israel and Stewart [1], and are identical to those found earlier in Refs. [5, 6]. Furthermore, these equations are formally similar to the relaxation equations of a single-component system but feature different transport coefficients, which contain the microscopic interactions of all components. For a single-component fluid, *i.e.* for $N_{\rm spec} = N_q = 1$, the results of Ref. [7] are identically reproduced.

Here, the first-order transport coefficients of the mixture are: the bulk viscosity ζ, the diffusion coefficients $\kappa_{qq'}$, and the shear viscosity η. These are obtained by summing over all particle species

$$\zeta \equiv -\sum_{s=1}^{N_{\rm spec}} \frac{m_s^2}{3}\zeta_{s,0}\,, \qquad \eta \equiv \sum_{s=1}^{N_{\rm spec}} \eta_{s,0}\,, \qquad \kappa_{qq'} \equiv \sum_{s=1}^{N_{\rm spec}} q_s \kappa_{s,0,q'}\,. \tag{16}$$

The relaxation time of the bulk viscos pressure τ_Π, the diffusion currents, $\tau_{qq'}$, and the stress tensor τ_π are related to the inverse of the linearized collision term (2), while all the remaining second-order transport coefficients are listed in Ref. [2].

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