NOVEL RELAXATION TIME APPROXIMATION: A CONSISTENT CALCULATION OF TRANSPORT COEFFICIENTS WITH QCD-INSPIRED RELAXATION TIMES*

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We use a novel formulation of the relaxation time approximation to consistently calculate the bulk and shear viscosity coefficients using QCD-inspired energy-dependent relaxation times and phenomenological thermal masses obtained from fits to lattice QCD thermodynamics. The matching conditions are conveniently chosen to simplify the computations.

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1. Introduction

Nuclear matter in extreme conditions can be investigated through ultra-relativistic heavy-ion collisions. In particular, obtaining the transport coefficients of the quark–gluon plasma, throughout the QCD phase diagram, is a very challenging task that is currently beyond the reach of first-principles techniques [1]. In this contribution, we compute the transport coefficients of an effective kinetic model [2, 3] with a temperature-dependent mass whose

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equation of state mimics lattice QCD thermodynamics [4]. We use the new relaxation time approximation (RTA) of the relativistic Boltzmann equation proposed in [5] and impose alternative matching conditions such that the interaction energy [6] depends only on the temperature even out of equilibrium.

2. The quasi-particle model

The relativistic Boltzmann equation for quasi-particles with a temperature-dependent mass, $M(T)$, is given by [7]

$$p^\mu \partial_\mu f_{\boldsymbol{p}} + \tfrac{1}{2}\partial_i M^2(T)\partial^i_{(\boldsymbol{p})} f_{\boldsymbol{p}} = C\left[f_{\boldsymbol{p}}\right] , \tag{1}$$

where $f_{\boldsymbol{p}} = f(x, \boldsymbol{p})$ is the single-particle distribution function. Above, $\partial^i_{(\boldsymbol{p})} = \partial/\partial \boldsymbol{p}_i$, and $C\left[f_{\boldsymbol{p}}\right]$ is the collision integral.

In the limit of vanishing net-charge, the main dynamical equation is the continuity equation for the energy-momentum tensor, $T^{\mu\nu}$,

$$\partial_\mu T^{\mu\nu} = 0 . \tag{2}$$

In the presence of a thermal mass, $T^{\mu\nu} \equiv \langle p^\mu p^\nu \rangle + g^{\mu\nu} B$, where B is the interaction energy [6], $g^{\mu\nu}$ denotes the metric, $\langle \cdots \rangle = \int \mathrm{d}P \cdots f_{\boldsymbol{p}}$, $\int \mathrm{d}P = g \int \mathrm{d}^3\boldsymbol{p}/[(2\pi)^3 E_{\boldsymbol{p}}]$, with g being the degeneracy factor and $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + M^2}$. The interaction energy B satisfies the following dynamical equation:

$$\partial_\mu B = -\tfrac{1}{2}\partial_\mu M^2 \langle 1 \rangle , \tag{3}$$

which is valid both in and out of equilibrium. We consider the Maxwell–Boltzmann statistics so that in equilibrium $f_{\boldsymbol{p}} = \exp\left(-\beta u_\mu p^\mu\right) \equiv f_{0\boldsymbol{p}}$, with $\beta = 1/T$ and u_μ being the fluid 4-velocity (which satisfies $u_\mu u^\mu = 1$).

The temperature dependence of the mass is obtained such that the equation of state of the model describes lattice QCD results [4]. Plots for $B(T)$ and $M(T)$ can be seen in Refs. [2, 3]. Qualitatively, $M(T)/T$ is very large at low temperatures and saturates at $M(T)/T \approx 1.1$ at high temperatures.

3. Matching conditions and the collision term

The energy-momentum tensor can be decomposed in terms of the 4-velocity u^μ as follows:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} , \tag{4}$$

where ε is the total energy density, P is the total isotropic pressure, h^μ is the energy diffusion, $\pi^{\mu\nu}$ is the shear-stress tensor, and we defined the projection

operator $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. They are obtained from moments of $f_{\boldsymbol{p}}$ as explained in [7]. In general, ε_0 and P_0 may have non-equilibrium corrections, such that $\varepsilon = \varepsilon_0 + \delta\varepsilon$, $P = P_0 + \Pi$, respectively.

The meaning of u^μ and β for non-equilibrium states is determined by matching conditions [7]. The most widely used prescription is the one introduced by Landau [8], where $\delta\varepsilon \equiv 0$ and $h^\mu \equiv 0$. In the present work, we choose a new prescription in order to simplify Eq. (3). Specifically, we impose

$$\langle 1 \rangle \equiv \langle 1 \rangle_0 \,, \tag{5}$$

where $\langle \cdots \rangle_0 \equiv \int \mathrm{d}P \cdots f_{0\boldsymbol{p}}$, which defines the temperature for non-equilibrium states. In this matching, $\delta\varepsilon \neq 0$. To define the 4-velocity, a further condition is needed. However, since we only consider a fluid at vanishing chemical potential, our results will not depend on this particular choice.

With prescription (5), the interaction energy can be determined solely as a function of T, and Eq. (3) can be solved as if the system were in equilibrium,

$$\frac{\partial B(T)}{\partial T} = -\frac{gTM^2}{2\pi^2} K_1\left(\frac{M(T)}{T}\right) \frac{\partial M(T)}{\partial T} \,, \tag{6}$$

which can be readily integrated since $M(T)$ is known, and the boundary condition $B(0) = 0$ is given. Above, K_1 is the first modified Bessel function of the second kind.

Novel relaxation time approximation

In contrast to the traditional RTA [9], in the new prescription proposed in Ref. [5] the conservation laws hold at the microscopic level even when considering momentum-dependent relaxation times and arbitrary matching conditions. In practice, we approximate the collision term as [3]

$$C[f_{\boldsymbol{p}}] \approx -\frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} f_{0\boldsymbol{p}} \left[\phi_{\boldsymbol{p}} - \frac{\left\langle \phi_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}^2}{\tau_{\mathrm{R}}} \right\rangle_0}{\left\langle \frac{E_{\boldsymbol{p}}^3}{\tau_{\mathrm{R}}} \right\rangle_0} E_{\boldsymbol{p}} - \frac{\left\langle \phi_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} p^{\langle\mu\rangle} \right\rangle_0}{\frac{1}{3}\left\langle \Delta^{\alpha\beta} p_\alpha p_\beta \frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} \right\rangle_0} p_{\langle\mu\rangle} \right] \,, \tag{7}$$

where $\phi_{\boldsymbol{p}} \equiv (f_{\boldsymbol{p}} - f_{0\boldsymbol{p}})/f_{0\boldsymbol{p}}$. We parametrize the energy dependence of the relaxation time as $\tau_{\mathrm{R}} = t_{\mathrm{R}} \, (E_{\boldsymbol{p}}/T)^\gamma$, where the parameter γ encodes the information of the underlying microscopic interaction, and $t_{\mathrm{R}} > 0$. For instance, it has been argued that $\gamma = 1/2$ in QCD effective theories [10]. Above, we defined the space-like projection $p^{\langle\mu\rangle} = \Delta^{\mu\nu} p_\nu$.

4. Transport coefficients

In first-order theories, equation (2) is complemented by constitutive relations for the non-equilibrium currents ($\delta\varepsilon$, Π, $\pi^{\mu\nu}$). In kinetic theory, they can be calculated using the Chapman–Enskog expansion [7] which, when truncated at first order, leads to the following relativistic Navier–Stokes formulation of hydrodynamics:

$$\delta\varepsilon = \chi\theta\,, \qquad \Pi = -\zeta\theta\,, \qquad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}\,. \tag{8}$$

Using (7), the transport coefficients read [3]

$$\zeta = -\frac{1}{3}\left\langle (\Delta^{\mu\nu}p_\mu p_\nu)\, A_{\boldsymbol{p}}\frac{\tau_{\rm R}}{E_{\boldsymbol{p}}}\right\rangle_0 - \left\langle \frac{\tau_{\rm R}}{E_{\boldsymbol{p}}}A_{\boldsymbol{p}}\right\rangle_0 \frac{I_{3,1}}{I_{1,0}}\,, \tag{9}$$

$$\chi = -\langle A_{\boldsymbol{p}}\tau_{\rm R}E_{\boldsymbol{p}}\rangle_0 + \left\langle \frac{\tau_{\rm R}}{E_{\boldsymbol{p}}}A_{\boldsymbol{p}}\right\rangle_0 \frac{I_{3,0}}{I_{1,0}}\,, \qquad \eta = \frac{\beta}{15}\left\langle (\Delta^{\mu\nu}p_\mu p_\nu)^2\,\frac{\tau_{\rm R}}{E_{\boldsymbol{p}}}\right\rangle_0\,, \tag{10}$$

where $A_{\boldsymbol{p}} = -\beta c_{\rm s}^2 E_{\boldsymbol{p}}^2 - \frac{\beta}{3}\Delta^{\lambda\sigma}p_\lambda p_\sigma - \beta^2 M\frac{\partial M}{\partial\beta}c_{\rm s}^2$, and $c_{\rm s}^2 \equiv (\partial P_0/\partial\varepsilon_0) = (1/\beta)(I_{10}+I_{21})/[I_{30}+\frac{1}{2}I_{10}(\partial M^2/\partial\beta)\beta]$ is the speed of sound squared, which is expressed in terms of $I_{nq} = 1/[(2q+1)!!]\left\langle \left(-\Delta^{\lambda\sigma}p_\lambda p_\sigma\right)^q E_{\boldsymbol{p}}^{n-2q}\right\rangle_0$.

In Fig. 1, we plot the coefficients as functions of temperature for different values of the parameter γ, as well as the temperature dependence of the mass. For all values of γ investigated, $\zeta \geq 0$ and $\chi \leq 0$. In both figures, it is seen that the absolute values of the coefficients grow with γ. At low temperatures, where the effective mass is large, $M/T \to \infty$, all three normalized coefficients behave as $(M/T)^{\gamma-1}$. For $\gamma = 1$, $\eta = t_{\rm R}(\varepsilon_0 + P_0)$ at all temperatures. In the opposite limit, $M/T \to 0$, $\zeta = -(1/3)\chi \propto M(T)(\mathrm{d}/\mathrm{d}T)\,(M(T)/T)$, and $\eta \sim \Gamma(\gamma+5)/120$ [1].

Entropy production

The entropy current for classical quasiparticles is $S^\mu = \int \mathrm{d}P p^\mu f_{\boldsymbol{p}}(1 - \ln f_{\boldsymbol{p}})$. We note that the entropy production does not depend on the choice of matching conditions [3]. To first order in the Chapman–Enskog expansion, one finds

$$\partial_\mu S^\mu \simeq \zeta_{\rm s}\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu}\,, \qquad \zeta_{\rm s} = \left\langle \frac{\tau_{\rm R}}{E_{\boldsymbol{p}}}[A_{\boldsymbol{p}}]^2\right\rangle_0 = \zeta + c_{\rm s}^2\chi\,. \tag{11}$$

Since both $\zeta_{\rm s}$ and η are non-negative, so is the entropy production. The coefficient $\zeta_{\rm s}$ can be used to provide a matching-invariant interpretation of

[1] Even though this is not achieved at high temperatures, where $M/T \to 1.1$ [3], these expansions serve as estimates.

bulk viscosity and, indeed, for the Landau matching conditions $\zeta_s = \zeta$. This coefficient behaves similarly to ζ as a function of temperature, with the difference that, as $M/T \to 0$, $\zeta_s \propto [M(T)(\mathrm{d}/\mathrm{d}T)(M(T)/T)]^2$, thus displaying a steeper descent at high temperatures in Fig. 1.

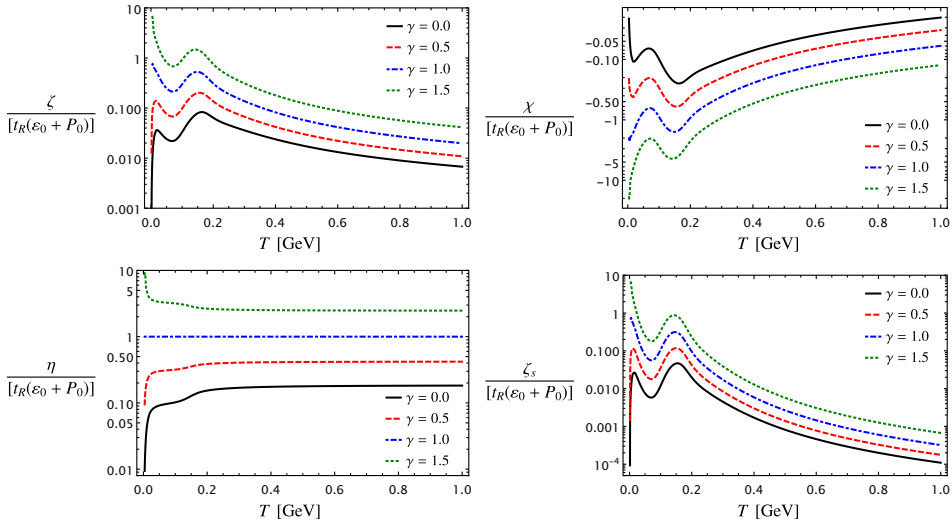

Fig. 1. (Top left) Normalized bulk viscosity, (top right) energy correction coefficient, (bottom left) shear viscosity, and (bottom right) matching-invariant bulk viscosity coefficients as functions of temperature. Each transport coefficient is shown for various values of the parameter γ.

5. Conclusion

In this work, we have computed the first-order transport coefficients of an effective kinetic model with temperature-dependent mass, using the new relaxation time approximation proposed in Ref. [5]. We have used an alternative matching condition [Eq. (5)] to simplify the computations, which in turn imply that there are nonzero out-of-equilibrium corrections to the energy density. We find that all transport coefficients are significantly affected by the choice of the parameter γ, which defines how the relaxation time depends on energy. Consistency with the second law of thermodynamics is demonstrated and used to derive a matching-invariant bulk viscosity coefficient. In future work, we intend to compute the transport coefficients that appear in other theories of hydrodynamics [11, 12] using the present model.

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