NOVEL RELAXATION TIME APPROXIMATION: A CONSISTENT CALCULATION OF TRANSPORT COEFFICIENTS WITH QCD-INSPIRED RELAXATION TIMES*

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We use a novel formulation of the relaxation time approximation to consistently calculate the bulk and shear viscosity coefficients using QCDinspired energy-dependent relaxation times and phenomenological thermal masses obtained from fits to lattice QCD thermodynamics. The matching conditions are conveniently chosen to simplify the computations.

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1. Introduction

Nuclear matter in extreme conditions can be investigated through ultrarelativistic heavy-ion collisions. In particular, obtaining the transport coefficients of the quark–gluon plasma, throughout the QCD phase diagram, is a very challenging task that is currently beyond the reach of first-principles techniques [1]. In this contribution, we compute the transport coefficients of an effective kinetic model [2, 3] with a temperature-dependent mass whose

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equation of state mimics lattice QCD thermodynamics [4]. We use the new relaxation time approximation (RTA) of the relativistic Boltzmann equation proposed in [5] and impose alternative matching conditions such that the interaction energy [6] depends only on the temperature even out of equilibrium.

2. The quasi-particle model

The relativistic Boltzmann equation for quasi-particles with a temperature-dependent mass, M(T), is given by [7]

$$p^{\mu}\partial_{\mu}f_{\boldsymbol{p}} + \frac{1}{2}\partial_{i}M^{2}(T)\partial^{i}_{(\boldsymbol{p})}f_{\boldsymbol{p}} = C\left[f_{\boldsymbol{p}}\right],\qquad(1)$$

where $f_{\mathbf{p}} = f(x, \mathbf{p})$ is the single-particle distribution function. Above, $\partial^i_{(\mathbf{p})} = \partial/\partial \mathbf{p}_i$, and $C[f_{\mathbf{p}}]$ is the collision integral.

In the limit of vanishing net-charge, the main dynamical equation is the continuity equation for the energy-momentum tensor, $T^{\mu\nu}$,

$$\partial_{\mu}T^{\mu\nu} = 0. \tag{2}$$

In the presence of a thermal mass, $T^{\mu\nu} \equiv \langle p^{\mu}p^{\nu} \rangle + g^{\mu\nu}B$, where *B* is the interaction energy [6], $g^{\mu\nu}$ denotes the metric, $\langle \cdots \rangle = \int dP \cdots f_{\mathbf{p}}, \int dP = g \int d^3 \mathbf{p} / [(2\pi)^3 E_{\mathbf{p}}]$, with *g* being the degeneracy factor and $E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + M^2}$. The interaction energy *B* satisfies the following dynamical equation:

$$\partial_{\mu}B = -\frac{1}{2}\partial_{\mu}M^2 \langle 1 \rangle , \qquad (3)$$

which is valid both in and out of equilibrium. We consider the Maxwell–Boltzmann statistics so that in equilibrium $f_{\mathbf{p}} = \exp(-\beta u_{\mu}p^{\mu}) \equiv f_{0\mathbf{p}}$, with $\beta = 1/T$ and u_{μ} being the fluid 4-velocity (which satisfies $u_{\mu}u^{\mu} = 1$).

The temperature dependence of the mass is obtained such that the equation of state of the model describes lattice QCD results [4]. Plots for B(T)and M(T) can be seen in Refs. [2, 3]. Qualitatively, M(T)/T is very large at low temperatures and saturates at $M(T)/T \approx 1.1$ at high temperatures.

3. Matching conditions and the collision term

The energy-momentum tensor can be decomposed in terms of the 4-velocity u^{μ} as follows:

$$T^{\mu\nu} = \varepsilon u^{\mu} u^{\nu} - P \Delta^{\mu\nu} + h^{\mu} u^{\nu} + h^{\nu} u^{\mu} + \pi^{\mu\nu} , \qquad (4)$$

where ε is the total energy density, P is the total isotropic pressure, h^{μ} is the energy diffusion, $\pi^{\mu\nu}$ is the shear-stress tensor, and we defined the projection

operator $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu}u^{\nu}$. They are obtained from moments of $f_{\mathbf{p}}$ as explained in [7]. In general, ε_0 and P_0 may have non-equilibrium corrections, such that $\varepsilon = \varepsilon_0 + \delta \varepsilon$, $P = P_0 + \Pi$, respectively.

The meaning of u^{μ} and β for non-equilibrium states is determined by matching conditions [7]. The most widely used prescription is the one introduced by Landau [8], where $\delta \varepsilon \equiv 0$ and $h^{\mu} \equiv 0$. In the present work, we choose a new prescription in order to simplify Eq. (3). Specifically, we impose

$$\langle 1 \rangle \equiv \langle 1 \rangle_0 , \qquad (5)$$

where $\langle \cdots \rangle_0 \equiv \int dP \cdots f_{0p}$, which defines the temperature for non-equilibrium states. In this matching, $\delta \varepsilon \neq 0$. To define the 4-velocity, a further condition is needed. However, since we only consider a fluid at vanishing chemical potential, our results will not depend on this particular choice.

With prescription (5), the interaction energy can be determined solely as a function of T, and Eq. (3) can be solved as if the system were in equilibrium,

$$\frac{\partial B(T)}{\partial T} = -\frac{gTM^2}{2\pi^2} K_1\left(\frac{M(T)}{T}\right) \frac{\partial M(T)}{\partial T},\qquad(6)$$

which can be readily integrated since M(T) is known, and the boundary condition B(0) = 0 is given. Above, K_1 is the first modified Bessel function of the second kind.

Novel relaxation time approximation

In contrast to the traditional RTA [9], in the new prescription proposed in Ref. [5] the conservation laws hold at the microscopic level even when considering momentum-dependent relaxation times and arbitrary matching conditions. In practice, we approximate the collision term as [3]

$$C[f_{\boldsymbol{p}}] \approx -\frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} f_{0\boldsymbol{p}} \left[\phi_{\boldsymbol{p}} - \frac{\left\langle \phi_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}^{2}}{\tau_{\mathrm{R}}} \right\rangle_{0}}{\left\langle \frac{E_{\boldsymbol{p}}^{2}}{\tau_{\mathrm{R}}} \right\rangle_{0}} E_{\boldsymbol{p}} - \frac{\left\langle \phi_{\boldsymbol{p}} \frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} p^{\langle \mu \rangle} \right\rangle_{0}}{\frac{1}{3} \left\langle \Delta^{\alpha\beta} p_{\alpha} p_{\beta} \frac{E_{\boldsymbol{p}}}{\tau_{\mathrm{R}}} \right\rangle_{0}} p_{\langle \mu \rangle} \right] , \quad (7)$$

where $\phi_{\mathbf{p}} \equiv (f_{\mathbf{p}} - f_{0\mathbf{p}})/f_{0\mathbf{p}}$. We parametrize the energy dependence of the relaxation time as $\tau_{\mathrm{R}} = t_{\mathrm{R}} (E_{\mathbf{p}}/T)^{\gamma}$, where the parameter γ encodes the information of the underlying microscopic interaction, and $t_{\mathrm{R}} > 0$. For instance, it has been argued that $\gamma = 1/2$ in QCD effective theories [10]. Above, we defined the space-like projection $p^{\langle \mu \rangle} = \Delta^{\mu\nu} p_{\nu}$.

4. Transport coefficients

In first-order theories, equation (2) is complemented by constitutive relations for the non-equilibrium currents ($\delta \varepsilon$, Π , $\pi^{\mu\nu}$). In kinetic theory, they can be calculated using the Chapman–Enskog expansion [7] which, when truncated at first order, leads to the following relativistic Navier–Stokes formulation of hydrodynamics:

$$\delta \varepsilon = \chi \theta$$
, $\Pi = -\zeta \theta$, $\pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}$. (8)

Using (7), the transport coefficients read [3]

$$\zeta = -\frac{1}{3} \left\langle \left(\Delta^{\mu\nu} p_{\mu} p_{\nu} \right) A_{\boldsymbol{p}} \frac{\tau_{\mathrm{R}}}{E_{\boldsymbol{p}}} \right\rangle_{0} - \left\langle \frac{\tau_{\mathrm{R}}}{E_{\boldsymbol{p}}} A_{\boldsymbol{p}} \right\rangle_{0} \frac{I_{3,1}}{I_{1,0}}, \tag{9}$$

$$\chi = -\left\langle A_{\boldsymbol{p}}\tau_{\mathrm{R}}E_{\boldsymbol{p}}\right\rangle_{0} + \left\langle \frac{\tau_{\mathrm{R}}}{E_{\boldsymbol{p}}}A_{\boldsymbol{p}}\right\rangle_{0}\frac{I_{3,0}}{I_{1,0}}, \qquad \eta = \frac{\beta}{15}\left\langle \left(\Delta^{\mu\nu}p_{\mu}p_{\nu}\right)^{2}\frac{\tau_{\mathrm{R}}}{E_{\boldsymbol{p}}}\right\rangle_{0}, (10)$$

where $A_{\mathbf{p}} = -\beta c_{\mathrm{s}}^2 E_{\mathbf{p}}^2 - \frac{\beta}{3} \Delta^{\lambda \sigma} p_{\lambda} p_{\sigma} - \beta^2 M \frac{\partial M}{\partial \beta} c_{\mathrm{s}}^2$, and $c_{\mathrm{s}}^2 \equiv (\partial P_0 / \partial \varepsilon_0) = (1/\beta)(I_{10} + I_{21})/[I_{30} + \frac{1}{2}I_{10}(\partial M^2 / \partial \beta)\beta]$ is the speed of sound squared, which is expressed in terms of $I_{nq} = 1/[(2q+1)!!] \left\langle \left(-\Delta^{\lambda \sigma} p_{\lambda} p_{\sigma}\right)^q E_{\mathbf{p}}^{n-2q} \right\rangle_0$.

In Fig. 1, we plot the coefficients as functions of temperature for different values of the parameter γ , as well as the temperature dependence of the mass. For all values of γ investigated, $\zeta \geq 0$ and $\chi \leq 0$. In both figures, it is seen that the absolute values of the coefficients grow with γ . At low temperatures, where the effective mass is large, $M/T \to \infty$, all three normalized coefficients behave as $(M/T)^{\gamma-1}$. For $\gamma = 1$, $\eta = t_{\rm R}(\varepsilon_0 + P_0)$ at all temperatures. In the opposite limit, $M/T \to 0$, $\zeta = -(1/3)\chi \propto M(T)(d/dT) (M(T)/T)$, and $\eta \sim \Gamma(\gamma + 5)/120^{-1}$.

Entropy production

The entropy current for classical quasiparticles is $S^{\mu} = \int dP p^{\mu} f_{\mathbf{p}}(1 - \ln f_{\mathbf{p}})$. We note that the entropy production does not depend on the choice of matching conditions [3]. To first order in the Chapman–Enskog expansion, one finds

$$\partial_{\mu}S^{\mu} \simeq \zeta_{\rm s}\theta^2 + 2\eta\sigma^{\mu\nu}\sigma_{\mu\nu}, \qquad \zeta_{\rm s} = \left\langle \frac{\tau_{\rm R}}{E_{\boldsymbol{p}}} [A_{\boldsymbol{p}}]^2 \right\rangle_0 = \zeta + c_{\rm s}^2\chi.$$
 (11)

Since both ζ_s and η are non-negative, so is the entropy production. The coefficient ζ_s can be used to provide a matching-invariant interpretation of

 $^{^1}$ Even though this is not achieved at high temperatures, where $M/T \rightarrow 1.1$ [3], these expansions serve as estimates.

bulk viscosity and, indeed, for the Landau matching conditions $\zeta_s = \zeta$. This coefficient behaves similarly to ζ as a function of temperature, with the difference that, as $M/T \to 0$, $\zeta_s \propto [M(T)(d/dT)(M(T)/T)]^2$, thus displaying a steeper descent at high temperatures in Fig. 1.



Fig. 1. (Top left) Normalized bulk viscosity, (top right) energy correction coefficient, (bottom left) shear viscosity, and (bottom right) matching-invariant bulk viscosity coefficients as functions of temperature. Each transport coefficient is shown for various values of the parameter γ .

5. Conclusion

In this work, we have computed the first-order transport coefficients of an effective kinetic model with temperature-dependent mass, using the new relaxation time approximation proposed in Ref. [5]. We have used an alternative matching condition [Eq. (5)] to simplify the computations, which in turn imply that there are nonzero out-of-equilibrium corrections to the energy density. We find that all transport coefficients are significantly affected by the choice of the parameter γ , which defines how the relaxation time depends on energy. Consistency with the second law of thermodynamics is demonstrated and used to derive a matching-invariant bulk viscosity coefficient. In future work, we intend to compute the transport coefficients that appear in other theories of hydrodynamics [11, 12] using the present model. G.S.R. is financially funded by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), process No. 142548/2019-7. G.S.D. also acknowledges CNPq as well as Fundação Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ), process No. E-26/202.747/ 2018. M.N.F is supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) grants 2017/05685-2 and 2020/12795-1. J.N. is partially supported by the U.S. Department of Energy, Office of Science, Office for Nuclear Physics under Award No. DE-SC0021301.

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