INITIAL STATE AND APPROACH TO EQUILIBRIUM*

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A possible resolution of the early thermalisation puzzle is provided by the notion of far-from-equilibrium attractors which arise due to the specific kinematics of heavy-ion collisions. Attractors appear in a wide variety of dynamical models and it is plausible that they also occur in QCD. The physical implications of these observations depend on how robust this effect is when typically made symmetry restrictions are relaxed. I briefly review this line of research and its perspectives.

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1. Introduction

Quark–gluon plasma created in heavy-ion collision experiments is initially in a highly complex nonequilibrium state, while by the time when hadrons appear, it is thermal to a large degree. Much effort is being devoted to understanding which features of the initial state are imprinted on experimentally accessible observables and how this happens. A key role in the picture which has emerged is played by models formulated in the language of fluid dynamics. It is very natural to use such a description close to local equilibrium, but hydrodynamic simulations are initialized at much earlier times when the system is very anisotropic. The successful application of hydrodynamic models in such far-from-equilibrium situations implies that the complexity of initial states is rapidly reduced within an interval of proper time shorter than 1 fm following the collision. Since this happens for all initial states, the system can be said to reach a far-from-equilibrium *hydrodynamic attractor* in a process referred to as hydrodynamisation. These words have been given a fairly precise meaning in many models describing conformal Bjorken flow.

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Attractors occur in a number of situations in physics, such as in cosmology where the inflationary attractor can be linked with the interplay between Hubble expansion and the inflaton potential gradient. In models of conformal Bjorken flow, far-from-equilibrium attractors are a consequence of the special kinematics characteristic of ultrarelativistic heavy-ion collisions. In Ref. [1], this was succinctly phrased by saying "the main features of the dynamics of expanding plasmas are determined by the competition between the expansion itself, which is dictated by the external conditions of the collisions, and the collisions among the plasma constituents which generically tend to isotropize the particle momentum distribution functions". Recently, this point was amplified in Refs. [2, 3], where these two regimes were clearly distinguishable. The importance of recognising the kinematic origin of far-from-equilibrium attractors is that it explains their ubiquity. It suggests that the success of hydrodynamics in this context might not signal the need to revise the fundamentals of fluid mechanics, but instead could be a consequence of specific kinematical circumstances. The experience gained in studies of toy models may be directly relevant to the real-world problem, provided we understand how robust are the features of boost-invariant attractors when some symmetry requirements are relaxed. Ultimately, one would hope that a suitable hydrodynamic model may succeed in capturing essential features of the QCD attractor, thanks to its reduced complexity [4].

2. Conformal Bjorken flow

The symmetries of Bjorken flow (which originate in the ultrarelativistic nature of the collision) are boost invariance along the collision axis and invariance under rotations and translations in the transverse plane. If in addition we assume conformal symmetry so that the energy-momentum tensor is traceless, its expectation value takes the form of

$$T^{\mu}_{\nu} = \mathrm{diag}\left\{-\mathcal{E}(\tau), \mathcal{P}_{\mathrm{L}}(\tau), \mathcal{P}_{\mathrm{T}}(\tau), \mathcal{P}_{\mathrm{T}}(\tau)\right\} \,, \tag{1}$$

where τ is the proper time and

$$\mathcal{P}_{\mathrm{L}} \equiv \frac{\mathcal{E}}{3}\left(1 - \frac{2}{3}\mathcal{A}\right) \,, \qquad \mathcal{P}_{\mathrm{T}} \equiv \frac{\mathcal{E}}{3}\left(1 + \frac{1}{3}\mathcal{A}\right) \,. \tag{2}$$

Here, $\mathcal{E}(\tau)$ is the energy density and $\mathcal{A}(\tau)$ is the pressure anisotropy, which is a measure of distance from equilibrium. Introducing the off-equilibrium effective temperature by $\mathcal{E} \sim T^4$, it is very convenient to use dimensionless variables $(\mathcal{A}, w \equiv \tau T)$. In terms of these, the conservation of energy-momentum can be written as

$$\frac{\mathrm{d}\log T}{\mathrm{d}\log w} = \frac{\mathcal{A} - 6}{\mathcal{A} + 12} \,. \tag{3}$$

This differential equation can be trivially integrated once the function $\mathcal{A}(w)$ is given; it determines the solution up to a single integration constant. In this way, the problem is reduced to determining $\mathcal{A}(w)$. For a perfect fluid, $\mathcal{A}=0$ and the solution is Bjorken's $T \sim \tau^{-1/3}$.

Depending on the dynamical model, $\mathcal{A}(w)$ may involve complicated initial data, which is dissipated away in the course of equilibration, since asymptotically all solutions approach perfect fluid behaviour. The basic observation is that in many models, there is a special "attractor solution" which is approached by all other solutions even when the system is still very anisotropic. To the extent that $\mathcal{A}(w)$ can be approximated by the attractor, the only remnant of the initial state is the integration constant arising from Eq. (3), which sets the overall energy scale of a given event.

3. Modelling the QCD attractor

The paradigmatic example of a hydrodynamic attractor appears in the MIS model, where the pressure anisotropy $\mathcal{A}(w)$ satisfies a first-order ODE

$$C_{\tau\Pi}\left(1+\frac{\mathcal{A}}{12}\right)\mathcal{A}' + \left(\frac{C_{\tau\Pi}}{3w}+\frac{C_{\lambda_1}}{8C_\eta}\right)\mathcal{A}^2 = \frac{3}{2}\left(\frac{8C_\eta}{w}-\mathcal{A}\right), \tag{4}$$

where $C_\eta, C_{\tau\Pi}$, and C_{λ_1} are dimensionless transport coefficients. There is a unique solution, denoted below by $\mathcal{A}_\star$, which is regular at $w=0$

$$\mathcal{A}_\star(w) = 6\sqrt{\frac{C_\eta}{C_{\tau\Pi}}} + O(w) \tag{5}$$

and acts as an attractor. At early times, generic solutions approach it as $\mathcal{A}-\mathcal{A}_\star \sim w^{-4}$. This behaviour is independent of the transport coefficients, which suggests that it is a kinematical effect due to the longitudinal expansion. It is also significant that solutions whose pressure anisotropy is below the attractor at early times are initially driven *away* from equilibrium.

The regular value of the pressure anisotropy at $w=0$ is related to the behaviour of the energy density at early times by the relation

$$\mathcal{E} \sim \tau^{-\beta} \iff \mathcal{A}_\star(0) = 6\left(1-\frac{3\beta}{4}\right) \tag{6}$$

so that the attractor is free-streaming if $\mathcal{A}_\star(0)=3/2$. This can be imposed by a choice of transport coefficients in Eq. (5), but is not required in general.

At late times, the asymptotic behaviour of any solution depends on the dynamics through the transport coefficients

$$\mathcal{A}(w) = \frac{8C_\eta}{w} + \frac{16C_\eta(C_{\tau\Pi}-C_{\lambda_1})}{3w^2} + O\left(\frac{1}{w^3}\right) \tag{7}$$

but it is independent of initial conditions up to exponentially-damped corrections. In reality, the deconfinement transition is reached before the system is fully in the asymptotic regime, so some such dependence on initial conditions should be present even in this idealized setting.

The series appearing in Eq. (7) can be interpreted as the hydrodynamic gradient expansion and has a vanishing radius of convergence, which is connected with the dissipative nature of the system. This property has recently been shown to hold for a large class of more general flows [5].

Far-from-equilibrium attractors are a typical feature of conformal Bjorken flow also in more general hydrodynamic models [6, 7]. Such models are constructed to reproduce the asymptotics of microscopic theories near equilibrium; their solutions coincide only in the late-time asymptotic region (see *e.g.* Ref. [8]). Some models are more complex than MIS, but may capture more information about the initial state. They may also try to mimic nontrivial nonhydrodynamic sectors appearing in microscopic theories or match the asymptotics to higher orders in gradients. Finally, they may also alleviate issues with causality violations discussed recently in Refs. [9, 10].

In hydrodynamic models of conformal Bjorken flow, the attractor $\mathcal{A}_\star(w)$ is a particular solution regular at the origin, and the relevant initial condition can be determined directly from the evolution equations. Identifying attractors in kinetic theory models is less direct. Early work considered collision kernels in the relaxation-time approximation (RTA), but recently attractors have been identified in a more realistic kinetic theory model involving the AMY collision kernel [11]. Other recent studies of attractors in kinetic theory include Refs. [12–16]. An important common feature of these attractors is that they are free-streaming at early times.

The strongly coupled $\mathcal{N} = 4$ supersymmetric Yang–Mills theory has been an important theoretical laboratory for studies of hydrodynamisation thanks to the AdS/CFT correspondence. The late-time behaviour of its hydrodynamic attractor, set by the shear viscosity, can be extended to intermediate times by Borel summation of the gradient expansion [17] but its form at early times is less certain [2]. Clarifying this issue is a very interesting topic for further study. The problem is made more difficult by the high dimensionality of the relevant phase space, so the projection of the dynamics onto the $(w, \mathcal{A})$ plane may be misleading. A simple illustration of such a situation appears in a much simpler context in Ref. [6].

4. Attractors and the initial state

Early-time attractors aim to provide a bridge between models of the initial energy deposition and hydrodynamic simulations. Given the partial loss of information as the attractor is approached, a key question is which features of the initial state survive so as to be accessible to measurement. A possible description of prehydrodynamic evolution is provided by free streaming (see *e.g.* Ref. [18]). Free-streaming attractors provide a simple picture of hydrodynamisation which has recently been shown to be consistent with experiment assuming a specific model of the initial state [19]. However, while free streaming is a feature of early-time attractors in kinetic theory, it is not necessarily so in general. In particular, the early-time behaviour of attractors in hydrodynamic models is determined by the transport coefficients and need not be tuned to free streaming. In a recent study, it was shown that consistency with the experiment requires that the early-time behaviour of the attractor must match the model of the initial state [20].

5. Beyond conformal Bjorken flow

Up to this point, we have reviewed various aspects of conformal Bjorken flow, emphasizing the decisive role of longitudinal dynamics in the process of hydrodynamisation. The key question is: How robust are intuitions gleaned from such toy models once symmetry restrictions are lifted? The most important simplifications which need to be assessed are conformal invariance and suppression of transverse dynamics. As soon as any symmetry restrictions are relaxed, the system has more degrees of freedom. A significant issue which arises is finding an advantageous choice of variables which would make attractor behaviour manifest.

Attractors in Bjorken flow without conformal symmetry were investigated in Refs. [21–23]. These articles focused on a particular nonconformal model of kinetic theory where the conformal symmetry is broken due to quasiparticles of nonvanishing mass. It was found that early-time attractors arise only in specific combinations of dissipative currents. It was also found that the standard effective hydrodynamic description is able to capture this behaviour only if it is suitably modified. One would expect further studies aimed at clarifying how hydrodynamic models can capture early-time attractors in such nonconformal cases.

If the longitudinal expansion is dominant at early times, the attractor may retain its relevance even in the presence of transverse dynamics. The persistence of early-time attractors in such circumstances was studied in Ref. [2] in the case of kinetic theory in the RTA. It was found that with sufficiently early initialisation, nontrivial transverse profiles had negligible effect on the early-time behaviour, with the attractor governing early-time dynamics. Only at late times were the effects of transverse structure visible. Effects of transverse dynamics were also the subject of Ref. [24], where kinetic theory in the RTA was compared with hydrodynamics and a transport model (BAMPS).

6. Attractors in phase space

In the case of conformal Bjorken flow, we have seen that universal variables $(\mathcal{A}, w)$ exist which are correlated even far from equilibrium, making attractor behaviour manifest. Once some symmetry restrictions are relaxed, the number of degrees of freedom increases and it is not known how to identify such variables in general. An approach to this problem was formulated in Ref. [3]. It tracks the behaviour of solutions on slices of phase space at constant proper time. If one starts out with a set of initial conditions spanning a D-dimensional region on the initial time slice, these solutions end up in a region of lower dimensionality $d < D$ on slices at later times. The attractor phenomenon can thus be identified with this reduction of dimensionality of sets of solutions, as exemplified by the plots in Fig. 1. This framing of the problem makes it amenable to exploration using techniques

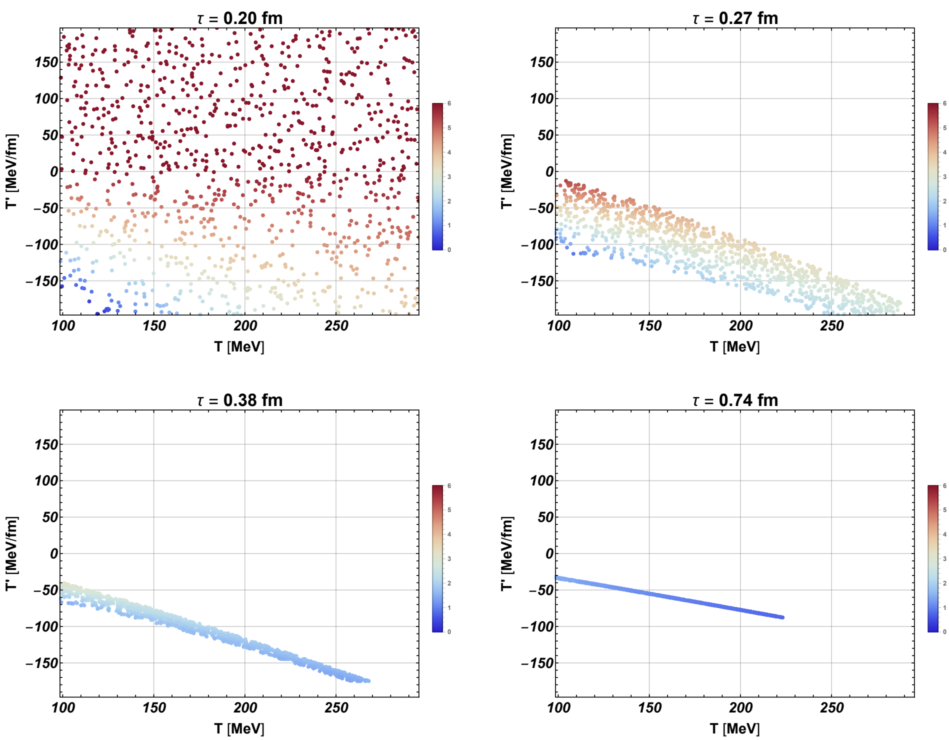

Fig. 1. (Colour on-line) A sequence of snapshots expressing the evolution of a point-cloud of solutions plotted on a proper-time slice in boost-invariant MIS theory. Initially, the depicted region is uniformly filled, but in subsequent plots, we see the dimensionality reduced from 2 to 1. The colour of a dot encodes the pressure anisotropy.

of machine learning. Initially, this approach was tested only in some cases of Bjorken flow in hydrodynamic models and a model of kinetic theory in the RTA [3]. In this analysis, the early, expansion-dominated phase was clearly visible and quantified using Principal Component Analysis. More recently, this kind of methodology was applied to Bjorken flow in the context of kinetic theory with the AMY kernel [25]. There appear to be no fundamental obstructions to applying it to general flows, since no special parameterisation of phase space is required.

7. Summary

Approximate boost-invariance at early times may be the key element of the early thermalisation puzzle, since it leads to far-from-equilibrium attractor behaviour identified in diverse dynamical settings which share the kinematic features characteristic of heavy-ion collisions. Recent studies suggest that such attractors exist also when some of the idealisations present in toy models are relaxed. However, new approaches will be needed to identify and make use of attractors in such situations due to the greater number of degrees of freedom.

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