NUCLEAR EQUATION OF STATE FROM NUCLEAR COLLECTIVE EXCITED STATE PROPERTIES*

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In this work, I present a personal overview of recent investigations of overall properties in Giant Resonances that impact our understanding of the nuclear Equation of State around saturation density.

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1. Introduction

Giant Resonances are collective excitations in nuclei that have been known for several decades [1]. The excitation energy and strength distribution of such resonances depend on the underlying nuclear interaction and have been very useful in characterizing the nuclear Equation of State (EoS) [2] — a definition and short introduction are given in Sec. 2. Specially important is the role of sum rules (Sec. 3) which, in some special cases, allow for direct access to basic nuclear properties. In this respect, one of the most paradigmatic examples is the incompressibility of the finite nucleus [3].

In this contribution, I will briefly review a selection of recent analyses of the Giant Monopole (GMR), Dipole (GDR) resonances, in Secs. 4 and 5 respectively, as well as the Isobaric Analog Resonance (IAS) and the Spin-Dipole Resonance (SDR) in Secs. 6 and 7, respectively. These sections will be complemented with an introductory discussion of sum rules using, in some cases, simple models to justify the physical meaning and soundness of the presented theoretical analysis.

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2. The nuclear Equation of State

The nuclear Equation of State (EoS) is commonly defined as the energy per particle ($e \equiv E/A$) of an unpolarized infinite system of neutrons and protons at zero temperature and where the Coulomb interaction is neglected. Hence, it is customarily written in terms of the neutron and proton densities $e(\rho_n, \rho_p)$ or, equivalently, in terms of the total density $\rho \equiv \rho_n + \rho_p$ and relative difference $\beta \equiv (\rho_n + \rho_p)/\rho$ as $e(\rho, \beta)$. Although this is an ideal system, it is very useful in order to theoretically investigate the impact of nuclear interaction on the bulk properties of nuclei and neutron stars.

Stable nuclei typically show small values of β. Due to this and assuming isospin symmetry, it is useful to expand $e(\rho, \beta)$ for small β as

$$e(\rho, \beta) = e(\rho, 0) + S(\rho)\beta^2 + O\left[\beta^4\right] , \tag{1}$$

where it has been shown that around saturation, the parabolic expansion is already a very good approximation even for $\beta = 1$. The first term in the right-hand side of the equation is the so-called symmetric matter EoS, while the second term is the so-called symmetry energy, a penalty energy to the system for departing from the most stable configuration $e(\rho, 0)$.

Finite nuclei and some properties of neutron stars are sensitive to densities around the nuclear saturation density ($\rho_0 = 0.16$ fm^{-3}). Expanding the symmetric matter EoS and the symmetry energy around ρ_0 allows to define different coefficients that would characterize the EoS and can be *easily* calculated with most of the nuclear models available in the literature. That is,

$$e(\rho, 0) = e(\rho_0, 0) + \frac{1}{2}K\epsilon^2 + O\left[\epsilon^3\right] , \tag{2}$$

$$S(\rho) = J - L\epsilon + \frac{1}{2}K_{\text{sym}}\epsilon^2 + O\left[\epsilon^3\right] , \tag{3}$$

where $\epsilon \equiv (\rho_0 - \rho)/3\rho_0$.

In the present contribution, I will present a discussion on how the values of K, J, and L have been — or could be — estimated from the theoretical analysis of the experimental data on the isoscalar (IS) GMR, isovector (IV) GDR as well as on two charge exchange resonances: the IAS and the SDR.

3. Strength function and sum rules

The reaction-independent part of the nuclear response to an external perturbation is encoded in the strength function

$$S(E) \equiv \sum_{\nu} |\langle\nu|\mathcal{O}|0\rangle|^2 \delta(E - E_\nu - E_0) , \tag{4}$$

where $\mathcal{O}$ is the transition operator that models the specific excitation proved in experiment, $|0\rangle$ is the ground state, and $|\nu\rangle$ an excited state.

Of special interest for this contribution are the moments of the strength function, also referred to in the literature as sum rules

$$m_k = \int \mathrm{d}E E^k S(E) = \sum_\nu |\langle \nu|\mathcal{O}|0\rangle|^2 \delta(E_\nu - E_0)^k \,. \tag{5}$$

Assuming the completeness of the excitation spectra, one can rewrite the sum rules in a computationally convenient way involving only an expectation value on the ground state. Of particular interest for the study of Giant Resonances are the non-energy-weighted sum rule

$$m_0 = \langle 0|\mathcal{O}^\dagger \mathcal{O}|0\rangle = \frac{1}{2}\langle 0| \left\{\mathcal{O}^\dagger, \mathcal{O}\right\} |0\rangle \,, \tag{6}$$

and the energy-weighted sum rule

$$m_1 = \frac{1}{2}\langle 0| \left[\mathcal{O}^\dagger, [\mathcal{H}, \mathcal{O}]\right] |0\rangle \,, \tag{7}$$

as well as the inverse energy-weighted sum rule. The latter can be calculated by means of the dielectric theorem that relates the variation of the expectation value of the Hamiltonian under the action of a small perturbation $\lambda\mathcal{O}$ with the inverse energy-weighted sum rule. In perturbation theory,

$$\delta\langle\mathcal{H}\rangle = \lambda^2 \sum_\nu \frac{|\langle \nu|\mathcal{O}|0\rangle|^2}{E_\nu - E_0} + O\left[\lambda^3\right] = \lambda^2 m_{-1} + O\left[\lambda^3\right] \,, \tag{8}$$

where a variation in the expectation value of the operator can be written as

$$\delta\langle\mathcal{O}\rangle = -2\lambda \sum_\nu \frac{|\langle \nu|\mathcal{O}|0\rangle|^2}{E_\nu - E_0} + O\left[\lambda^2\right] = -2m_{-1} + O\left[\lambda^2\right] \tag{9}$$

and, thus,

$$m_{-1} = \frac{1}{2}\left.\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\lambda^2}\right|_{\lambda=0} = -\frac{1}{2}\left.\frac{\partial\langle\mathcal{O}\rangle}{\partial\lambda}\right|_{\lambda=0} \tag{10}$$

or, equivalently,

$$\frac{1}{m_{-1}} = -2\frac{\partial^2\langle\mathcal{H}\rangle}{\partial\langle\mathcal{O}\rangle^2} \,. \tag{11}$$

Based on these sum rules, one can define two different excitation energies (E_x) of a Giant Resonance, the centroid and constrained E_x are defined as

$$E_x^{\mathrm{cent}} \equiv \frac{m_1}{m_0} \quad \text{and} \quad E_x^{\mathrm{cons}} \equiv \sqrt{\frac{m_1}{m_{-1}}} \,. \tag{12}$$

4. Giant Monopole Resonance

The ISGMR is a collective mode associated with nuclear excitations with a change in the angular orbital momentum $\Delta L = 0$ and the spin $\Delta S = 0$. The theoretical operator that models monopole transitions is

$$\mathcal{O}_{\mathrm{GMR}} = \sum_i^A r^2 Y_{00}(\hat{r})\,. \tag{13}$$

Assuming this operator, one can write

$$\left(E_x^{\mathrm{ISGMR}}\right)^2 = \frac{m_1}{m_{-1}} = 4\frac{\hbar^2}{m}\left\langle r^2\right\rangle \frac{\partial^2 E}{\partial\langle r^2\rangle^2} \equiv K_A \frac{\hbar^2}{m\langle r^2\rangle}\,, \tag{14}$$

where the incompressibility of a finite nucleus K_A has been defined in an analogous way to the thermodynamic definition of the inverse of compressibility. That is [2],

$$K_A \equiv 4\left\langle r^2\right\rangle^2 \frac{\partial^2 E}{\partial\langle r^2\rangle^2}\,, \tag{15}$$

hence, a theoretical proof of the relation of E_x^{ISGMR} with incompressibility of the infinite system $K = K_{A\to\infty}$. Based on this insight, many works have analyzed the experimental data on E_x^{ISGMR} in order to characterize K in $e(\rho, 0)$. Note that the analysis will always require a modeling of the surface and isospin-asymmetry effects.

One of the main problems that have been raised in the analysis of the ISGMR in connection with the K parameter of the EoS is that models that tend to describe E_x in closed-shell nuclei such as ^{208}Pb overestimate the E_x in open-shell nuclei such as Sn isotopes [3]. Different possibilities have been discussed in the literature (see Refs. [2, 3] for details). Here, I just briefly discuss one of the main and new recent results [4], where the softness of the Sn isotopes is addressed by advocating for correlations beyond the mean-field.

In Ref. [4], it is shown that pairing effects allow for a larger number of active configurations with respect to magic nuclei predicting a larger energy shift of the ISGMR when particle-vibrations effects are considered in open-shell Sn isotopes (*cf.* Fig. 2 in [4]). This feature paves the way to a unified description of the monopole resonance and to a coherent analysis that may shed some light on the value of K. In Fig. 1, the excitation energy of the ISGMR in ^{120}Sn (upper panel) and ^{48}Ca (lower panel) as a function of the excitation energy of the ISGMR in ^{208}Pb as predicted by a set of Skyrme functionals is shown. Black squares correspond to calculations based on the Quasi-particle Random Phase Approximation QRPA, while blue circles

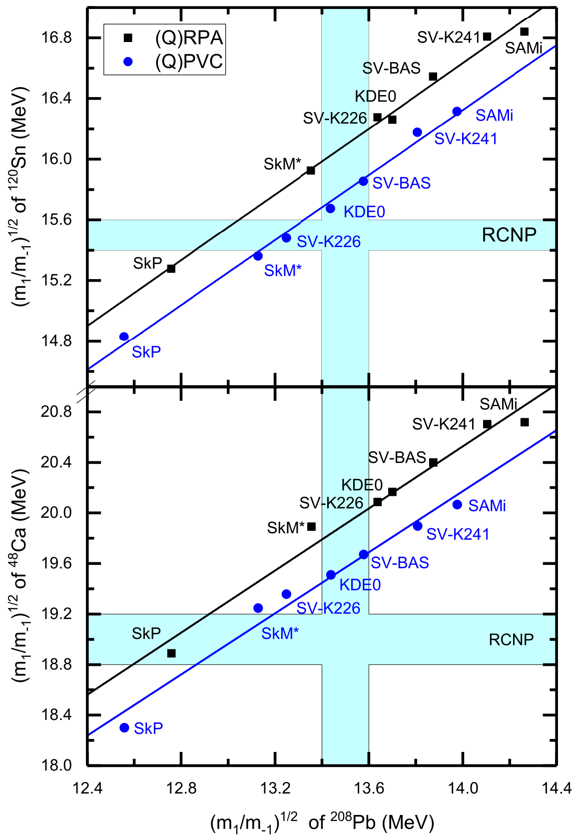

Fig. 1. (Colour on-line) Excitation energy of the ISGMR in ^{120}Sn (upper panel) and ^{48}Ca (lower panel) as a function of the excitation energy of the ISGMR in ^{208}Pb. Predictions with different Skyrme functionals are shown. Black squares correspond to calculations based on the Quasi-particle Random Phase Approximation, while blue circles correspond to the Quasi-particle Particle Vibration Coupling approach. Figure taken from Ref. [4].

correspond to the Quasi-particle Particle Vibration Coupling (QPVC) approach. Although the experimental crossing area (light blue bands) is not crossed by theory, the inclusion of correlations beyond the mean-field approach allows for a clear improvement. The best description is given by SV-K226 and KDE0 models, which are characterized by incompressibility values of 226 MeV and 229 MeV, respectively, at the mean-field level. Regarding the connection between beyond mean-field calculations using effective interactions fitted at the mean-field level — as in [4] — that would be instrumental to reliably determine K, some work is still needed.

5. Giant Dipole Resonance

The IVGDR is a collective mode associated with nuclear excitations with a change in the angular momentum $\Delta L = 0$ and spin $\Delta S = 0$. Since we are interested here on the isovector channel, protons and neutrons will be proven differently. Specifically, the theoretical operator that models isovector dipole transitions and that correctly subtracts effects coming from the center-of-mass motion is

$$\mathcal{O}_{\mathrm{GDR}} = \frac{N}{A}\sum_i^Z rY_{1M}(\hat{r}_i) - \frac{Z}{A}\sum_i^N rY_{1M}(\hat{r}_i)\,, \tag{16}$$

where one must average over the magnetic quantum number M. Assuming this operator, one can apply the Dielectric Theorem to calculate m_{-1} and, thus, the electric dipole polarizability $\alpha_D = (8\pi e^2/9)m_{-1}$. Another option is to rely on the RPA or other more complex many-body approximations, such as the PVC. In order to gain a simple physical insight into this observable, in Ref. [5], the Droplet Model (DM) expression was proposed for guidance (see [6] for more details)

$$\alpha_D^{\mathrm{DM}} \approx \frac{\pi e^2}{54}\frac{A\langle r^2\rangle}{J}\left(1 + \frac{5}{3}\frac{L}{J}\epsilon_{\mathrm{A}}\right)\,, \tag{17}$$

where $\epsilon_{\mathrm{A}} \equiv (\rho_0 - \rho_{\mathrm{A}})/s\rho_0$ and ρ_{A} is an average density probed in experiments measuring α_D and provided that this simple macroscopic approach captures the main features of the electric dipole polarizability (see a discussion in [2, 7] for more details). The latter equation points towards a correlation between $\alpha_D J$ and L that is fulfilled by nuclear Energy Density Functionals (EDFs) of the Skyrme and relativistic type as shown in Fig. 2 for the case of

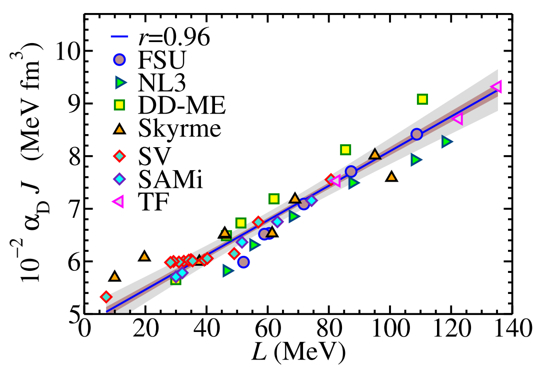

Fig. 2. Dipole polarizability in ^{208}Pb times the symmetry energy at saturation J as a function of the slope parameter L. Figure taken from Ref. [5].

^{208}Pb — RPA calculations have been used in this publication. This type of analysis based on EDFs has allowed to determine a linear relation between J and L on the basis of the experimental data on α_D [7]

$$
\begin{aligned}
J &= (24.5 \pm 0.8) + (0.168 \pm 0.007)L \quad \text{from } ^{208}\text{Pb}\,, && (18)\\
J &= (24.9 \pm 2.0) + (0.19 \pm 0.02)L \quad \text{from } ^{68}\text{Ni}\,, && (19)\\
J &= (25.4 \pm 1.1) + (0.17 \pm 0.01)L \quad \text{from } ^{120}\text{Sn}\,. && (20)
\end{aligned}
$$

6. Isobaric Analog State

The IAS is a collective mode associated with nuclear excitations with an isospin charge exchange. The theoretical operator that models this transitions is

$$\mathcal{O}^{\pm}_{\text{IAS}} = \sum_i^A t_{\pm}(i) \equiv T_{\pm}\,, \tag{21}$$

where $t_{\pm} \equiv \tau_{\pm}/2$ and $\tau_{\pm}$ are the Pauli matrices in isospin space that exchange a neutron into a proton and *vice versa* and follow the same commutation relations as the ladder operators. The excitation energy of this resonance, assuming now the τ_- channel which is dominant in a neutron rich nucleus, can then be calculated as

$$E_x^{\text{IAS}} = \frac{m_1}{m_0} = \frac{\langle 0|T_+[\mathcal{H}, T_-]|0\rangle}{\langle 0|T_+T_-|0\rangle}\,. \tag{22}$$

Hence, the only terms that contribute to E_x^{IAS} are those that break isospin symmetry ($[\mathcal{H}, T_-] \neq 0$). The largest term in the nuclear Hamiltonian that breaks isospin symmetry is the Coulomb potential, being the leading term, the Coulomb direct term (Hartree) with respect to the Coulomb exchange (Fock) and with respect to Coulomb corrections[1]. However, a small contribution from nuclear Isospin Symmetry Breaking (ISB) effects must be taken into account not only for a detailed study of the IAS and the Nolen—Schiffer anomaly [8] but also for the study of the nuclear EoS [9].

The excitation energy of the IAS can be related to the neutron skin thickness ($\Delta r_{np} = \langle r_n^2\rangle^{1/2} - \langle r_p^2\rangle^{1/2}$) and, thus, to the slope parameter L [10, 11] using a very simple model based on the fact that the Coulomb direct term will give the largest contribution to E_x^{IAS}. That is, assuming

[1] Those corrections are of different nature. Many-body correction to the first order Hartree–Fock scheme, as well as QED corrections to the Coulomb potential generated by protons. Electromagnetic spin–orbit contributions from protons and neutrons are small but would also need to be taken into account.

no isospin mixing ($\langle 0|T_+T_-|0\rangle = N - Z$) and a sharp sphere model for the neutron and proton densities, one finds

$$E_x^{\rm IAS} = \frac{6}{5}\frac{Ze^2}{R_p}\left(1 - \frac{1}{2}\frac{N}{N-Z}\frac{R_n - R_p}{R_p}\right) , \tag{23}$$

where $\Delta r_{np} = \sqrt{(3/5)}(R_n - R_p)$. This general trend of increasing $E_x^{\rm IAS}$ for decreasing neutron skin thickness is shown in Fig. 3 by actual RPA calculations based on Skyrme and covariant EDFs (see Ref. [9] for details on the calculations and the experimental constraints shown as black arrows). The models used in this figure contain only Coulomb direct and Coulomb exchange — in Slater approximation — contributions to the IAS energy and none of them is able to describe the experimental value (shown as a black dashed line). What is missing are Coulomb corrections plus some contribution from nuclear ISB terms, the latter being model-dependent and unknown in the nuclear medium. Hence, the determination of ISB in the medium can shed light on the EoS parameter L provided the fact that we know with exquisite accuracy the $E_x^{\rm IAS}$ in many nuclei.

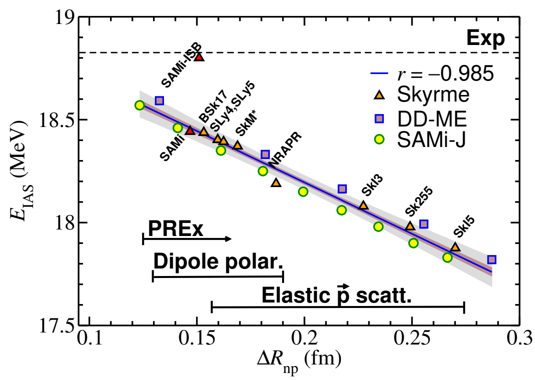

Fig. 3. Energy of the IAS as a function of the neutron skin thickness in ^{208}Pb. The arrows indicate the experimental results from polarized proton elastic scattering, parity-violating elastic electron scattering, and the electric dipole polarizability (see Ref. [9] for details). Figure taken from Ref. [9].

7. Spin-Dipole Resonance

The SDR is a collective charge exchange mode associated with nuclear excitations with a change in the angular orbital momentum $\Delta L = 1$ and the spin $\Delta S = 1$. The theoretical operator that modes spin-dipole transitions is

$$\mathcal{O}^{\pm}_{\rm SDR} = \sum_i^A \tau_\pm(i) r_i^L \left[Y_{1M}(\hat{r}_i) \otimes \boldsymbol{\sigma}(i)\right]_{JM} . \tag{24}$$

Theoretically, the connection of this type of nuclear excitations and the EoS is particularly simple in this mode. Indeed, the difference of the non-energy-weighted sum rules in the two isospin channels is

$$m_0\left(\mathcal{O}^-_{\mathrm{SDR}}\right) - m_0\left(\mathcal{O}^+_{\mathrm{SDR}}\right) = \frac{9}{4\pi}\left(N\left\langle r_n^2\right\rangle^{1/2} - Z\left\langle r_p^2\right\rangle^{1/2}\right) . \tag{25}$$

This expression can be written in terms of the charge radius — well known from experiment — and the neutron skin thickness of the nucleus under study. Hence, the SDR can give valuable hints on the slope parameter L. Different measurements are available in the literature [12]. We show in Fig. 4 the experimental results for the SDR in ^{90}Zr [13] (red dots) that agree well with theoretical calculations — Skyrme EDFs with and without tensor terms — and would predict via the present sum rule approach a neutron skin thickness in ^{90}Zr of 0.07 ± 0.04 fm. Data on the SDR in ^{208}Pb exist [14]. However, this experimental result would predict a neutron skin thickness in this nucleus that is very small as compared with other experimental and theoretical analyses.

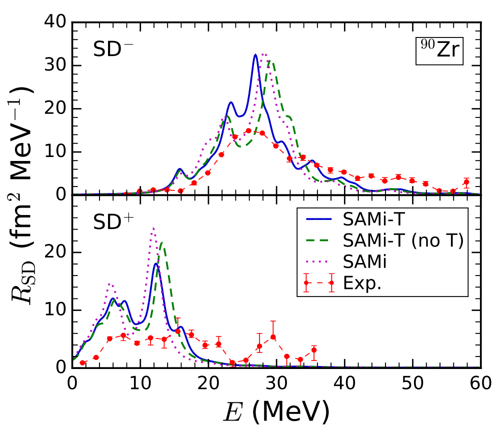

Fig. 4. (Colour on-line) SDR strength function of the τ_- (top) and τ_+ (bottom) channel in ^{90}Zr calculated by SAMi-T with and without tensor, in comparison with experimental and SAMi functional (see Ref. [15] for details on the calculations). Figure taken from Ref. [15].

8. Conclusions

Collective excitation modes have been used over the years to learn about the nuclear equation of state at around saturation density. Specifically, sum rules of some selected modes have been instrumental for this aim [2]. In this contribution, I have briefly presented some modes of collective excitation that are of particular interest to this topic and that, at the same time, have revitalized the field in the last years. Special emphasis must be given to the ISGMR and the IVGDR in this context since robust experimental data exist. With the advent of new experimental techniques, such type of studies are now a reality also in exotic nuclei. On the other side, solid theoretical methods as well as clear interpretations support recent analyses that connect all modes presented here with some basic parameters of the nuclear equation of state.

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