NEW DEVELOPMENTS IN $\mathcal{N} = 2$ SUPERSYMMETRIC GAUGE THEORIES: FROM INTEGRABILITY TO BLACK HOLES*

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We explain how new physical results in 4D $\mathcal{N} = 2$ supersymmetric gauge theories can be found by connecting them to 2D quantum integrable models. In particular, we set up an identification between the basic mathematical and physical objects of the two kinds of theories (the Q or Y and T functions of integrability and the two periods of the gauge theories) and then, we derive a stream of concepts and mathematical identities between them. Moreover, we use this new correspondence to prove, understand, and possibly generalise a recent application of gauge theories to black holes perturbation theory. From this, several new insights follow into black holes physics, especially a new powerful way of computing quasinormal mode frequencies (the Thermodynamic Bethe Ansatz nonlinear integral equation), characterising the gravitational wave signal (in the ringdown phase of black hole merging). For simplicity and limits of space, we restrict the discussion to the simplest case of the Liouville integrable model/pure SU(2) gauge theory/D3 brane gravitation background triad.

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1. Introduction

The Seiberg and Witten (SW) construction for $\mathcal{N} = 2$ SUSY gauge theories has the advantage of computing exactly several quantities [1]. In particular, the SW theory enjoys a weak–strong coupling duality which allows us to compute the full effective action for the light fields at any coupling. In practice, this theory prescribes computing the effective prepotential $\mathcal{F}^{(0)}$ by means of peculiar periods $a^{(0)}, a_{\mathrm{D}}^{(0)}$, defined as cycles (integrals) of a differential λ. The latter may be derived in turn from a hyperelliptic curve y_{SW}.

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In the subsequent decades, it was devised a procedure to compute instanton contributions to the prepotential $\mathcal{F}$. It requires a deformation of spacetime (called then Ω-background) through two complex parameters ϵ_1, ϵ_2. Then $\mathcal{F}$ can be computed order by order in the instanton (exponential) coupling Λ through combinatorial calculus on Young diagrams of the gauge group representations [2]. In the case of only one deformation ϵ_1 as $\epsilon_2 \to 0$, the so-called Nekrasov–Shatashvili (NS) limit [3], the SW differential becomes quantised in an ordinary differential equation (ODE). In the case of SU(2) gauge group, this is a (time-independent) Schrödinger equation in which $\epsilon_1 = \hbar$ plays the role of the Planck constant. The original classical SW hyperelliptic curve is recovered simply by the leading order of the WKB (asymptotic) expansion as $\hbar \to 0$. The quantum SW periods can be defined as the cycle integrals of quantum momentum $\mathcal{P}(y) = -i\hbar\frac{\mathrm{d}}{\mathrm{d}y}\ln\psi(y)$ of the solution ψ of the ODE [4]

$$\begin{pmatrix} a(\hbar, u, m, \Lambda_0) \\ a_{\mathrm{D}}(\hbar, u, m, \Lambda_0) \end{pmatrix} = \oint\limits_{A,B} \mathcal{P}(y, \hbar, u, m, \Lambda_0)\,\mathrm{d}y = 2\pi i \sum_n \mathrm{Res}\mathcal{P}(y)\bigg|_{y_n^{A,B}} . \quad (1)$$

We shall indicate by $a^{(n)}$, $a_{\mathrm{D}}^{(n)}$, $n \in \mathbb{N}$ their $\hbar \to 0$ asymptotic expansion modes.

Recently, the very same NS-deformed $\mathcal{N} = 2$ SU(2) gauge theories found new applications to black holes (BHs) perturbation theory [5–8]. It was found that quantizations conditions on quantum gauge periods a_{D}, a provide a new exact characterisation of quasinormal modes (QNMs) ω_n, which are the characteristic frequencies of the gravitational wave signal in the ring-down (final) phase of BHs merging [5]. From this, which was dubbed SW-QNM correspondence [8], many other applications and new results followed. Interestingly, the BHs which can be studied through this approach are also very "real" (for instance, the Kerr BHs) and enter astrophysics and gravitation phenomenology, and thus the experimental search for deviations from General Relativity (GR), such as horizon-scale structure [9, 10].

This paper is organized as follows. In Section 2, we explain a new kind of gauge-integrability correspondence for 4D $\mathcal{N} = 2$ SUSY in the NS limit and 2D integrable modes. In particular, we find basic identifications between Baxter's Q, Y, and T functions of integrability and the periods a, a_{D} of the gauge theories, and from this, new results on both sides of the correspondence follow. Then in Section 3, we use this correspondence to prove the SW-QNM correspondence. This allows us also to find through integrability a new method to compute QNMs of black holes. For limits of space, we only deal with the simplest case, the Liouville integrable model/pure SU(2) gauge theory/D3 brane gravitation background triad. For further details, explanations, extensions, and general validity of the method, we refer to our recent papers [11–17].

2. A new gauge/integrability correspondence

The quantum SW curve of pure ($N_\mathrm{f} = 0$) SU(2) $\mathcal{N} = 2$ SUSY in the NS limit is the following ODE:

$$-\frac{\hbar^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi(y) + \left[\Lambda_0^2 \cosh y + u\right]\psi(y) = 0\,. \tag{2}$$

Provided we set the change of variables as $\frac{\hbar}{\Lambda_0} = \mathrm{e}^{-\theta}$, $\frac{u}{\Lambda_0^2} = \frac{P^2}{2}\mathrm{e}^{-2\theta}$, it becomes the ODE for the self-dual (central charge $c = 25$) Liouville model [11]

$$\left\{-\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 2\,\mathrm{e}^{2\theta}\cosh y + P^2\right\}\psi(y) = 0\,. \tag{3}$$

The starting point of ODE/IM correspondence is finding the regular solutions at the singular points at $y \to \pm\infty$

$$\psi_{\pm,0}(y) \simeq 2^{-\frac{1}{2}}\mathrm{e}^{-\frac{1}{2}\theta\mp\frac{1}{4}y}\mathrm{e}^{-\mathrm{e}^{\theta\pm y/2}}\,, \qquad y \to \pm\infty\,. \tag{4}$$

Then we can use the discrete symmetries of the equation

$$\Omega_\pm : \quad y \to y \pm i\pi\,, \qquad \theta \to \theta + i\pi/2\,, \qquad P \to P \tag{5}$$

to define other independent solutions $\psi_{-,k} = \Omega_-^k\psi_{-,0}$, $\psi_{+,k} = \Omega_+^k\psi_{+,0}$, which have these invariance properties $\Omega_+^k\psi_{-,0} = \psi_{-,0}$, $\Omega_-^k\psi_{+,0} = \psi_{+,0}$. These solutions are normalized so that their Wronskians are $W[\psi_{-,k+1}, \psi_{-,k}] = -i$, $W[\psi_{+,k+1}, \psi_{+,k}] = i$. Now, we can define the Q and T functions (vacuum eigenvalue of corresponding operators) as the Wronskians of the regular solutions at different singular points or the same point in different Stokes sectors, respectively,

$$Q(\theta) = W[\psi_+, \psi_-]\,, \qquad T(\theta) = i\,W[\psi_{+,-1}, \psi_{+,1}] = -i\,W[\psi_{-,-1}, \psi_{-,1}]\,. \tag{6}$$

By the properties of Wronskians and connexion relations between solutions at different singular points or in different Stokes sectors (see [11, 15] for details), we can obtain the QQ and TQ systems

$$Q(\theta + i\pi/2)Q(\theta - i\pi/2) = 1 + Q(\theta)^2\,, \tag{7}$$

$$T(\theta)Q(\theta) = Q(\theta - i\pi/2) + Q(\theta + i\pi/2)\,. \tag{8}$$

We define also the Y function as $Y(\theta, P) = Q^2(\theta, P)$ and from (7) the Y-system follows

$$Y(\theta + i\pi/2)Y(\theta - i\pi/2) = (1 + Y(\theta))^2 \,. \tag{9}$$

Eventually, we solve it explicitly (*up to quadratures*) via a Thermodynamic Bethe Ansatz (TBA) integral equation for the pseudoenergy $\varepsilon(\theta) = -\ln Y(\theta)$

$$\varepsilon(\theta) = \frac{16\sqrt{\pi^3}}{\Gamma\left(\frac{1}{4}\right)^2}\,\mathrm{e}^{\theta} - 2\int_{-\infty}^{\infty} \frac{\ln\left[1 + \exp\{-\varepsilon(\theta')\}\right]}{\cosh(\theta - \theta')}\frac{\mathrm{d}\theta'}{2\pi} \,. \tag{10}$$

In this, P does not appear explicitly, but fixes the solution by defining the asymptotic linear behaviour as $\varepsilon(\theta, P) \simeq +8P\theta$, $P > 0$, at $\theta \to -\infty$ (which actually can be derived from the perturbative solution of ODE (3)).

In [11, 16] we proved that the quantum gauge periods a and a_{D} are directly connected to Baxter's Q and T functions as

$$Q(\theta, p) = \exp\left\{\frac{2\pi i}{\hbar}\, a_{\mathrm{D}}(\hbar, u, \Lambda_0)\right\} \,, \qquad T(\theta, p) = 2\cos\left\{\frac{2\pi}{\hbar} a(\hbar, u, \Lambda_0)\right\} \,. \tag{11}$$

As we have (11), we can search for a gauge interpretation of the integrability functional relations (7), (8), and (9). In particular, the QQ system gets the form of

$$1 + Q^2(\theta, u) = Q(\theta - i\pi/2, -u)Q(\theta + i\pi/2, -u) \,, \tag{12}$$

where we have considered that $\theta \to \theta \mp i\pi/2$ means $u \to -u$ (as P is fixed). Thus, it can be inverted into a TBA for the gauge period

$$\begin{aligned} \frac{2\pi i}{\hbar} a_{\mathrm{D}}(\hbar(\theta), -u, \Lambda_0) &= 2\pi i a_{\mathrm{D}}^{(0)}(-u, \Lambda_0)\frac{\mathrm{e}^{\theta}}{\Lambda_0} \\ &\quad + \int_{-\infty}^{\infty} \frac{\ln!\left[1 + \exp\left\{\frac{4\pi i}{\hbar} a_{\mathrm{D}}(\hbar(\theta'), u, \Lambda_0)\right\}\right]}{\cosh(\theta - \theta')}\frac{\mathrm{d}\theta'}{2\pi} \,, \end{aligned} \tag{13}$$

which eventually allows to compute exactly the prepotential $\mathcal{F}$. The TQ relation and T periodicity relations in the gauge variables read

$$T(\theta, u)Q(\theta, u) = Q(\theta - i\pi/2, -u) + Q(\theta + i\pi/2, -u) \,, \tag{14}$$

$$T(\theta, u) = T(\theta - i\pi/2, -u) \,, \tag{15}$$

and we can find out their meaning by checking that, with identifications (11) their asymptotic $\theta \to +\infty$ or $\hbar \to 0$ entail the $\mathbb{Z}_2$ symmetry relation for the other period extended to the NS-quantum theory [18, 19]

$$a_{\mathrm{D}}^{(n)}(-u) = i(-1)^n\left[-\operatorname{sgn}(\operatorname{Im} u)\, a_{\mathrm{D}}^{(n)}(u) + a^{(n)}(u)\right] \,, \tag{16}$$

$$a^{(n)}(-u) = -i(-1)^n \operatorname{sgn}(\operatorname{Im} u)\, a^{(n)}(u) \,. \tag{17}$$

Thus, relations (14) encode these $\mathbb{Z}_2$ relations among the asymptotic modes as unique exact equations, in a highly nonlinear and nontrivial way through the integrability structure. These were examples of new results for $\mathcal{N} = 2$ SUSY gained through the new identities (11) we proved, but from it, we can derive also new results for integrability, for instance, formulae of the integrals of motion in terms of the gauge periods (and *vice versa*) as explained in [11, 16].

3. A new gauge/integrability/gravity triality

Let us now illustrate briefly a further connexion between integrability and $\mathcal{N} = 2$ gauge theory to black holes perturbation theory, by taking the simple example of the D3 brane background. It has a line element

$$\mathrm{d}s^2 = H(r)^{-\frac{1}{2}}\left(-\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2\right) + H(r)^{\frac{1}{2}}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2\right), \tag{18}$$

where $\boldsymbol{x}$ are the longitudinal coordinates, $H(r) = 1 + L^4/r^4$, and $\mathrm{d}\Omega_5^2$ denotes the metric of the transverse round S^5-sphere. The scalar field perturbation in this background is [6]

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\phi(r) + \left[\omega^2\left(1 + \frac{L^4}{r^4}\right) - \frac{(l+2)^2 - \frac{1}{4}}{r^2}\right]\phi(r) = 0\,, \tag{19}$$

where ω is the QNM frequency and $l \in \mathbb{N}$. Upon the change of variables $r = L\mathrm{e}^{\frac{y}{2}}$, $\omega L = -2i\mathrm{e}^{\theta}$, $P = \frac{1}{2}(l+2)$, the equation reduces to equation (3) for the self-dual Liouville integrable model. As crucially noted in [15], the QNMs are defined as the zeroes of the same Wronksian (6) which defines the Q function (*cf.* [20]), namely the Bethe roots

$$Q(\theta_n) = 0\,, \tag{20}$$

which upon the gauge-integrability identification (11) directly implies quantization condition of the dual gauge period a_{D}

$$\frac{1}{\hbar}a_{\mathrm{D}}\left(\theta_n + \frac{i\pi}{2}, -u, \Lambda_0\right) = \frac{1}{2}\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{N} \tag{21}$$

as found heuristically in [5]. The QQ system (7) also characterizes the QNMs as $Y(\theta_n - i\pi/2) = -1$, from which the *TBA quantization condition* follows

$$\varepsilon(\theta_n - i\pi/2) = -i\pi(2n+1)\,, \qquad n \in \mathbb{Z}\,, \tag{22}$$

which can be easily implemented by using the TBA (10) as reported in Table 1. This constitutes a new method for computing QNMs, which may be convenient in some cases. The connexion to integrability is actually richer than what we can explain here, involving also sometimes the T function and showing an essential connexion with black hole physics, beyond the determination of QNMs [15–17].

Table 1. Comparison of QNMs of the D3 brane from the TBA (10) (through (22) with $n = 0$) and the Leaver method [21] (with $L = 1$).

n	l	TBA	Leaver
0	0	$\underline{1.369}12 - \underline{0.504}048i$	$\underline{1.369}72 - \underline{0.504}311i$
0	1	$\underline{2.091}18 - \underline{0.5017}88i$	$\underline{2.091}76 - \underline{0.5018}11i$
0	2	$\underline{2.80}57 - \underline{0.50100}9i$	$\underline{2.80}629 - \underline{0.50100}0i$
0	3	$\underline{3.517}23 - \underline{0.5006}49i$	$\underline{3.517}83 - \underline{0.5006}34i$
0	4	$\underline{4.227}28 - \underline{0.5004}53i$	$\underline{4.227}90 - \underline{0.5004}38i$

REFERENCES

[1] N. Seiberg, E. Witten, *Nucl. Phys. B* **426**, 19 (1994), arXiv:hep-th/9407087.

[2] N. Nekrasov, A. Okounkov, *Prog. Math.* **244**, 525 (2006), arXiv:hep-th/0306238.

[3] N.A. Nekrasov, S.L. Shatashvili, in: «XVI International Congress on Mathematical Physics», *World Scientific*, 2010, pp. 265–289, arXiv:0908.4052 [hep-th].

[4] A. Mironov, A. Morozov, *J. High Energy Phys.* **2010**, 040 (2010), arXiv:0910.5670 [hep-th].

[5] G. Aminov, A. Grassi, Y. Hatsuda, *Ann. Henri Poincaré* **23**, 1951 (2022), arXiv:2006.06111 [hep-th].

[6] M. Bianchi, D. Consoli, A. Grillo, J.F. Morales, *Phys. Lett. B* **824**, 136837 (2022), arXiv:2105.04245 [hep-th].

[7] G. Bonelli *et al.*, *Phys. Rev. D* **105**, 044047 (2022), arXiv:2105.04483 [hep-th].

[8] M. Bianchi *et al.*, *J. High Energy Phys.* **2022**, 024 (2022), arXiv:2109.09804 [hep-th].

[9] V. Cardoso, P. Pani, *Nature Astron.* **1**, 586 (2017), arXiv:1709.01525 [gr-qc].

[10] D.R. Mayerson, *Gen. Relativ. Gravitation* **52**, 115 (2020), arXiv:2010.09736 [hep-th].

[11] D. Fioravanti, D. Gregori, *Phys. Lett. B* **804**, 135376 (2020), arXiv:1908.08030 [hep-th].

[12] D. Fioravanti, H. Poghosyan, R. Poghossian, *J. High Energy Phys.* **2020**, 049 (2020), arXiv:1909.11100 [hep-th].

[13] D. Fioravanti, M. Rossi, H. Shu, *J. High Energy Phys.* **2020**, 086 (2020), arXiv:2004.10722 [hep-th].

[14] D. Fioravanti, M. Rossi, *Phys. Lett. B* **838**, 137706 (2023), arXiv:2106.07600 [hep-th].

[15] D. Fioravanti, D. Gregori, arXiv:2112.11434 [hep-th].

[16] D. Fioravanti, D. Gregori, H. Shu, arXiv:2208.14031 [hep-th].

[17] D. Gregori, D. Fioravanti, *PoS* **(ICHEP2022)**, 422 (2022).

[18] A. Bilal, F. Ferrari, *Nucl. Phys. B* **469**, 387 (1996), arXiv:hep-th/9602082.

[19] G. Basar, G.V. Dunne, *J. High Energy Phys.* **2015**, 160 (2015), arXiv:1501.05671 [hep-th].

[20] H.P. Nollert, *Class. Quantum Grav.* **16**, R159 (1999).

[21] E.W. Leaver, *Proc. R. Soc. Lond. A* **402**, 285 (1985).