CRITICAL RELAXATION IN ADS/CFT*

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Received 4 February 2023, accepted 18 April 2023,
published online 13 June 2023

It is not only known that hairy black holes can exist in asymptotically Anti-de Sitter (AdS) spaces, but also that in the context of the Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, such black holes can be interpreted as holographic duals of superfluids. After a perturbation, these black holes usually exhibit an exponentially damped ringing down described by quasi-normal modes, however, we will show that for perturbations around the exact critical point that characterizes the onset of the formation of scalar hair, this relaxation will exhibit a power law behaviour at late times. We will also explain how this can be interpreted through the lens of the AdS/CFT correspondence.

DOI:10.5506/APhysPolBSupp.16.6-A12

1. What is AdS/CFT?

As a concrete realisation of the more general holographic principle (see [1] for a review), the so-called *AdS/CFT correspondence* [2] posits that the

* Presented at the 8$^{\text{th}}$ Conference of the Polish Society on Relativity, Warsaw, Poland, 19–23 September 2022.

physics of certain conformal field theories (CFTs, referred to as the "boundary") can be equivalently encoded in the gravitational physics of a higher dimensional Anti-de Sitter space (AdS, referred to as the "bulk"), see Fig. 1 for illustration.

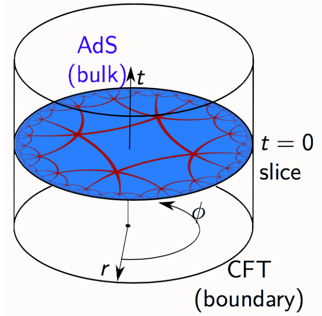

Fig. 1. Illustration of the AdS/CFT correspondence, here in the case of a three-dimensional bulk (with coordinates t, ϕ, r) and a two-dimensional boundary. As AdS space is negatively curved, the "boundary" is only an asymptotic boundary. In the chosen coordinate system, the $t = 0$ slice of the spacetime is mapped to a Poincaré disk.

The importance of the AdS/CFT correspondence is that it is a powerful concept that allows translating questions concerning one of the sides into the language of the other, where a new perspective on the problem can be gained, or where different mathematical methods may be at one's disposal. For example, AdS/CFT has been used as a tool to investigate out-of-equilibrium physics in the strongly coupled regime [3].

Here, we will be concerned specifically with a class of models commonly referred to as *holographic superconductors*, first introduced in a series of papers [4–6]. In the case of a four-dimensional bulk, the typical action of such a model reads [4–6]

$$S = S_{\text{grav}} + \frac{1}{2\kappa^2}\int \mathrm{d}x^4\,\sqrt{-g}\left[-\frac{1}{4\,q^2}F_{\mu\nu}F^{\mu\nu} - |D\varphi|^2 - m^2|\varphi|^2\right]. \qquad (1)$$

Here, a massive complex scalar field φ charged under a U(1) gauge field A_μ ($F = dA$) is coupled to an Einstein–Hilbert action (S_{grav}) with a negative cosmological constant. The latter is necessary for AdS/CFT to ensure that the negatively curved Anti-de Sitter spacetime is the appropriate vacuum.

While there are many technical details that could be explored in this context, the basic idea is as follows:

As its Hawking temperature is lowered, the AdS–Reissner–Nordström black hole becomes unstable against the formation of scalar hair via a second-order phase transition that takes place at a critical temperature T_c. *I.e.*, for $T < T_\mathrm{c}$, there will be a new stable solution with $\varphi \neq 0$. As we have explained above, in the context of the AdS/CFT correspondence we are invited to interpret everything that happens in the bulk gravity system also through the perspective of a boundary theory, and *vice versa*. So what is the boundary perspective on the formation of scalar black hole hair in the bulk? This corresponds to a spontaneous symmetry breaking of the U(1) symmetry by a scalar order parameter similar to what happens in a superconductor or superfluid, hence the nomenclature for this type of model. In fact, we can use the rules of the *holographic dictionary* derived in [2, 7, 8] to read off from the asymptotic expansion of the bulk fields φ and A_μ near infinity specific values corresponding to the expectation value of a complex order parameter $\langle \mathcal{O} \rangle$, a chemical potential which we call $\mathcal{A}_t$, and a charge density ρ of the putative superconductor model which is holographically dual to the bulk system according to the AdS/CFT correspondence. Likewise, the Hawking temperature of the black hole corresponds to the temperature of the dual field theory model.

Instead of changing the (Hawking) temperature and keeping other parameters constant, it is technically more convenient to keep the temperature constant[1] and change other parameters, such as the charge density ρ. The second-order phase transition to a phase with non-zero condensate (analogous to a superconducting phase) then happens at $\rho = \rho_\mathrm{c} \approx 4.06371$. See our publication [9] and citations therein for more details.

2. Critical relaxation in holographic superconductors

The goal of our paper [9], on which these proceedings are based, was to study how the holographic superconductor relaxes to its new equilibrium state after the parameter ρ is suddenly changed (also called *quenched*) to a value either exactly at $\rho = \rho_\mathrm{c}$ or close to it.

The motivation for this comes from the earlier paper [10], where a holographic model of the Kondo effect was studied, which shares many qualitative similarities with the holographic superconductors discussed above, including the existence of a second order-phase transition at which a scalar field in the bulk develops a non-zero value. We found there that in general, the relaxation of the system after a quench follows an exponential decay law (similar to so-called *quasi-normal-modes*, or QNMs), where the half-life time of the exponential decay diverges as the end state is brought close to the phase transition, a phenomenon well known as *critical slowing down* [11]. In-

[1] In fact, for the sake of simplicity, we keep the entire bulk metric fixed and only study the equations of motion of the matter fields on a fixed curved background spacetime for now. We hope to come back to the full problem including backreaction in the future.

terestingly, after exactly critical quenches, where formally the half-life time is infinite, we observed a power law decay of the modulus of the complex order parameter, with oscillations of its complex phase that were periodic on a *logarithmic time axis*. The latter phenomenon is associated with the emergence of a *discrete scale invariance*. This phenomenon has also been observed in a diverse set of physical systems [12], including the formation of black holes through critical collapse [13]. In fact, the discretely scale invariant complex phase oscillations found in [10] are particularly similar to the ones found in the critical collapse of a charged scalar field in [14].

Thus, our goal in [9] was to investigate whether, after exactly critical quenches, a similar discrete scale invariance would occur in the well-known holographic superconductor models, and whether a precise understanding of it could be provided mathematically.

After some extensive numerical simulations, we observed that indeed discrete scale invariance does appear in the relaxation after the exactly critical quenches: Writing the (now time-dependent) complex order parameter as (up to a trivial prefactor) $\langle \mathcal{O} \rangle \propto \Psi(t) = \phi(t)\, \mathrm{e}^{i\psi(t)}$, we found power law decays for the time-dependent quantities $\phi(t)$ and $\mathcal{A}(t) - \rho_c$, while the complex phase (which is of course only defined modulo 2π) $\psi(t)$ behaved as $\sim \log(t)$ — the expected signature of discrete scale invariance.

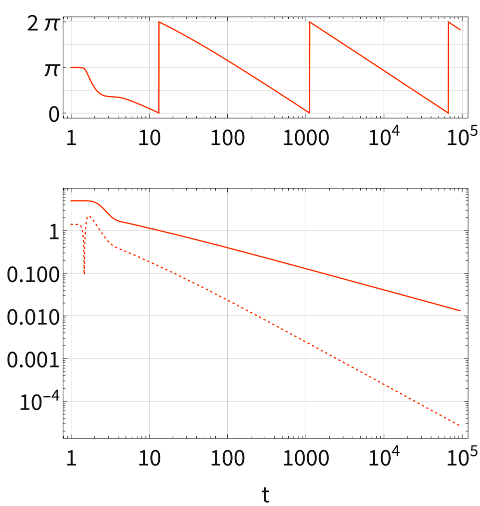

Fig. 2. Numerical results for $|\mathcal{A}_t(t) - \rho_c|$ (bottom frame, dashed curve), $|\langle \mathcal{O} \rangle| \equiv \phi(t)$ (bottom frame, solid curve), and the complex phase $\psi(t)$ (top frame) for an example of an exactly critical quench. Clearly, at late times, the complex phase rotates in a way that is periodic on the logarithmic time axis, a typical signal of discrete scale invariance [12]. The other curves show power law behaviours $\phi \propto 1/\sqrt{t}$ and $|\mathcal{A}_t - \rho_c| \propto 1/t$ at late times.

Specifically, the late-time behaviour after exactly critical quenches was well described by the model

$$\begin{aligned}\phi(t) &= A(t+\delta t)^{\alpha}\,, &(2)\\ \dot{\psi}(t)-(\mathcal{A}_t(t)-\rho_{\mathrm{c}}) &= B(t+\delta t)^{\gamma}\,, &(3)\end{aligned}$$

with fitted values

$$\begin{aligned}A &\approx 4.07\,, \qquad & \alpha &\approx -0.50\,,\\ B &\approx 0.93\,, \qquad & \gamma &\approx -1.00\,,\end{aligned} \tag{4}$$

and δt depending on the initial state before the quench. See Fig. 2 for illustration of representative findings.

3. Boundary model

As we further discussed in [9], when switching within the AdS/CFT correspondence to a boundary perspective, these observations can be explained very well by postulating a non-linear (complex) Ginzburg–Landau equation [15, 16] (see also [17, 18]) of the form

$$\begin{aligned}&\left[\partial_t - iC_1\left(\mathcal{A}_t(t)-\rho+C_5|\Psi(t)|^2\right)\right]\Psi(t)\\ &\equiv -(C_2+iC_3)\left[|\Psi(t)|^2-C_4(\rho-\rho_{\mathrm{c}})\right]\Psi(t)\,,\end{aligned} \tag{5}$$

where again $\Psi=\phi\,\mathrm{e}^{i\psi}$, $\rho,\rho_{\mathrm{c}},\psi,\mathcal{A}_t,C_i\in\mathbb{R}$, $\phi>0$. The parameter C_1 is the charge of the complex field which in our case is $+1$. As explained in detail in [9], the other parameters C_i can be fixed by comparing the predictions of (5) to the static solutions of the holographic superconductor and to the exponential falloff (at very late times) after near-critical quenches. The result is

$$\begin{aligned}C_2 &\approx 0.03018\,, \qquad & C_3 &\approx 0.09308\,,\\ C_4 &\approx 4.09192\,, \qquad & C_5 &\approx 0.14967\,.\end{aligned} \tag{6}$$

Once these parameters are fixed, equation (5) then allows us to make predictions for the behaviour of the holographic superconductor either for exactly critical quenches, or for the intermediate time behaviour of near-critical quenches. In fact, we can even provide simple analytic solutions for (5): In the exactly critical case (defined by $\rho=\rho_{\mathrm{c}}$), we obtain

$$\phi(t)=\frac{1}{\sqrt{2C_2t+\frac{1}{\phi(0)^2}}}\approx\frac{4.07}{t^{1/2}}+\dots\,, \tag{7}$$

and[2]

$$\dot{\psi} - C_1(\mathcal{A}_t - \rho_c) = \frac{C_1 C_5 + C_3}{2C_2 t + \frac{1}{\phi(0)^2}} \approx \frac{0.94}{t} + \dots \tag{8}$$

Clearly, this is in excellent agreement with our observations (2), (3), and (4). The solution (7) agrees with the scaling solution derived in [19] for an "initial-slip exponent" $\theta = 0$, which is indeed expected from the similar model [20].

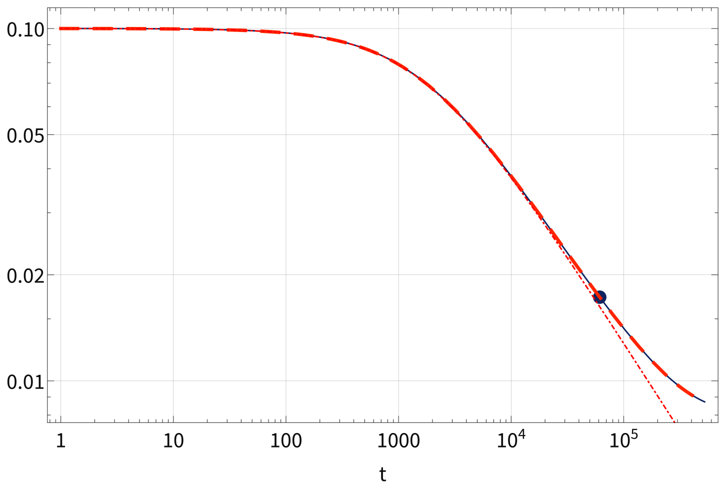

Fig. 3. Numerical (solid blue) and analytical (dashed orange, equation (9)) results for $\phi(t)$ after a near-critical quench, from ρ_{initial} =4.06626 to ρ_{final} =4.06373. Clearly, numerical and analytical curves agree very well. The dash-dotted red line shows the critical solution (7), which is expected to be a good approximation until the timescale t_{ho} which is signified by the (blue) dot.

Lastly, it is worth taking another look at the solutions of (5) for near-critical quenches, where ρ is extremely close but not identical to ρ_c. The solution then reads

$$\phi(t) = \sqrt{\frac{C_4(\rho - \rho_c)}{1 - \left(1 - \frac{C_4(\rho-\rho_c)}{\phi(0)^2}\right) e^{-2C_2 C_4 t(\rho-\rho_c)}}} \tag{9}$$

and another cumbersome expression for $\dot{\psi} - C_1\mathcal{A}_t$ is given in [9]. Interestingly, for $\rho - \rho_c \to 0$, this function describes a set of curves that exhibit a

[2] Equation (5) only allows to solve the combination $\dot{\psi} - C_1\mathcal{A}_t$ as a consequence of gauge invariance. Obtaining unique results for both $\psi(t)$ *and* $\mathcal{A}_t(t)$ would require the imposition of an additional gauge choice.

power-law falloff at intermediate times, which transitions to an exponential QNM-like falloff only at a timescale

$$t_{\text{ho}} \sim \frac{1}{|\rho - \rho_{\text{c}}|}\,. \tag{10}$$

In this sense, the power law falloff and discrete scale invariance that are characteristic for the exactly critical quenches are predicted to be observed also at intermediate times in the merely *near*-critical quenches. This is again clearly confirmed by our numerical simulations of the bulk dynamics. See Fig. 3 for illustration.

The work of M.F. was supported by the National Science Centre, Poland (NCN) grant 2021/42/E/ST2/00234. S.G. is supported by the U.S. Department of Energy, Office of Science grant No. DE-FG88ER41450. S.M. was supported through the grants CEX2020-001007-S and PGC2018-095976-B-C21 funded by MCIN/AEI/10.13039/501100011033 and by ERDF "A way of making Europe", as well as by an FPI-UAM predoctoral fellowship.

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