SPHERICAL AND NON-SPHERICAL COLLAPSE OF A FINITE TWO-LAYER BODY — INITIAL CONDITIONS*

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It is well-known that a collapsing single-layer sphere with spatially constant mass density leads to the Schwarzschild black hole, but the outcome is less clear when one considers the simplest multi-layer generalization of this: That of a homogeneous core surrounded by a homogeneous envelope of lower mass density, where both are joined at their common interface using the Darmois matching conditions. In this exploration, we set up the appropriate static initial conditions for the subsequent collapse of this two-layer sphere, and find that it is necessary to approximate the relative mass densities. We then go on to discuss how our setup could also be applied in the corresponding non-spherical case, where the ultimate goal is to determine at an analytical level how multiple layers may lead to the formation of something different to the Kerr black hole during a rotating collapse.

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1. Introduction

The final fate of gravitational collapse is one of the most pursued questions in General Relativity. The simplest configuration involving a sphere of pressureless matter with constant mass density was discovered by Oppenheimer and Snyder (hereafter denoted O–S) [1], where the end state is a Schwarzschild black hole. Lemaître, Tolman, and Bondi (LTB) [2], and many others have generalized this to investigate collapsing balls having spatially varying mass density, demonstrating that naked singularities can arise when the mass variation is significant. Price, Cunningham, and Moncrief [3] (denoted PCM) extended the homogeneous O–S collapse to include slow

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rotation; they showed that both Kerr and non-Kerr end states can arise, depending on the initial conditions. Numerical work on gravitational collapse under a variety of conditions is summarized in Joshi and Malafarina [4].

Rather than a single layer of constant mass density, in reality, astrophysical bodies are more likely to be composed of multiple layers with a higher mass density in the inner layers, and it is natural to ask whether (or not) the simplest multi-layer body having different spatially-constant mass densities in each layer will also collapse to a Schwarzschild black hole under all conditions, and then later explore the non-spherical PCM analog.

However, this multi-layer structure having a step-like distribution can be regarded as behaving in a somewhat analogous manner to a single-layer spheroid with a continuously varying mass density, where the matching conditions for the gravitational field at the mutual interface for our step-like distribution are expected to play a very important role.

Before we can make progress with departures from spherical symmetry however, we must first fully establish the spherical limit of such a collapsing configuration, which in this case, is likely to require some variant of the O–S evolution to be applied to each layer. Not only that, but we must also construct appropriate static initial conditions for this spherical limit.

There is already a wide variety of literature on spherical, static solutions (*e.g.* [5]), and further there has also been some work on exact solutions for static core–envelope structures, starting with Durgapal [6]. If we were *only* interested in the spherical case, then these could have been used as initial data for an LTB-like evolution. But we ultimately want to go beyond spherical symmetry, and unlike the spatially-constant mass density case, these other solutions and their respective LTB evolution do not easily extend to the non-spherical case.

More importantly, the single-layer O–S collapse requires a static sphere with a constant mass density at the start of the collapse [7]; this is the well-known Schwarzschild interior [8]. Therefore, an analogous requirement in our proposed spherical multi-layer collapse would be that we also start with a static body consisting of multiple layers, each with their own constant mass density.

With all this in mind, for the remainder of this paper, we shall focus our attention on the simplest possible generalization of the static single-layer sphere to a different sphere consisting of a constant mass density core, surrounded by an outer layer of (lower) constant mass density than the core. As we will later see, this is *not* the simple combination of two-single-layer metrics, because the outer layer will be shown to possess a highly non-trivial additional contribution to the field in order to satisfy the interface and boundary conditions. These are stipulated by the Darmois matching conditions of the induced metric and extrinsic curvature, which in the static case reduce to the matching of the pressure and the two metric components. In order to obtain our solution, we learn that it is necessary to make a new type of approximation, which is also likely to apply to the collapsing case in a future investigation.

2. Two-layer static interior

The general metric for a static sphere having isotropic pressure is given by

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu}\mathrm{d}t^2 + \mathrm{e}^{\lambda}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{1}$$

where Ω is the solid angle, and the independent variable r is defined by being orthogonal to the Killing hypersurfaces. Following a similar solution procedure to Stephani [9], the three field equations essentially amount to (where the dash denotes r differentiation, k is the gravitational constant, and μ is the mass density of the fluid): *(i)* Hydrostatic equilibrium

$$(p+\mu)' \;=\; -\frac{1}{2}\nu'(p+\mu)\,, \tag{2}$$

$$p+\mu \;=\; B\,\mathrm{e}^{-\frac{\nu}{2}} \tag{3}$$

and *(ii)* Conditions on the dr and dt coefficients respectively

$$kr^2\mu \;=\; -\left(\mathrm{e}^{-\lambda}r\right)' + 1\,, \tag{4}$$

$$\frac{\mathrm{e}^{-\lambda}}{r}\left(\lambda' + \mu'\right) \;=\; kB\,\mathrm{e}^{-\frac{\nu}{2}}\,, \tag{5}$$

where B is an integration constant of (4), and (3) is incorporated into the pressure term in the original field equation which led to (5). We want to apply this form of the field equations to our two-layer setup, which consists of an outer layer having (coordinate) radius (a) and mass density μ, and an inner core with radius (b) having mass density ρ. So far, no assumption is made about the relative magnitudes of the radii, or of the mass densities. Integrating (4) for the outer layer gives (where g is another integration constant)

$$\mathrm{e}^{-\lambda} \;=\; 1 - Ar^2 + \frac{g}{r}\,, \tag{6}$$

$$A \;=\; \frac{1}{3}k\mu\,. \tag{7}$$

One could argue that as the integration is over a layer of finite size, it must be evaluated as a definite integral with the limits (a) and (b) explicitly stated. Or, alternatively, one could initially evaluate it as an indefinite integral, and apply the boundary conditions which subsequently determine the integration constant g. Mathematically speaking, both methods are equally correct, but here we choose the latter approach in order to have as many integration parameters available as possible for when we eventually apply the mutual interface and external boundary conditions; the radial coordinate values corresponding to (a) and (b) would end up becoming a by-product of the matching conditions.

To deal with the field equation (5), use (6) and let $\mathrm{e}^{-\frac{\nu}{2}} = \sigma$. This gives us the following ODE, where transforming the derivative results in an extra factor of σ on the r.h.s.:

$$2\left(1 - Ar^2 + \frac{g}{r}\right)\sigma' - \left(2Ar + \frac{g}{r^2}\right)\sigma = kBr\sigma^2 \,. \tag{8}$$

Equation (8) is a Bernoulli-type equation where the r.h.s. is non-linear in σ; it can be converted to the following linear first-order ODE by the substitution $\sigma = z^{-1}$:

$$2\left(1 - Ar^2 + \frac{g}{r}\right)\frac{\mathrm{d}z}{\mathrm{d}r} + \left(2Ar + \frac{g}{r^2}\right)z = rkB \,. \tag{9}$$

As things currently stand, this equation cannot be solved in terms of closed functions if $g \neq 0$ [10]; $g = 0$ is the well-known Schwarzschild interior for the single-layer case. To get around the issue where $g \neq 0$, we now make two different approximations.

3. Small variation in the mass densities

The most obvious assumption to make is that the total gravitating mass of the core is only slightly larger than that of the outer layer; this can be achieved if the mass density of the core is slightly larger than the mass density of the outer layer. Correspondingly, this would require that the field in the outer layer is a small increment on top of what the single-layer case already provides. Therefore, letting $\varepsilon = \frac{g}{A}$ and $B \to B_0 + \varepsilon B_1$, Eq. (9) can be written as

$$2\left(1 - Ar^2 + \frac{A\varepsilon}{r}\right)\frac{\mathrm{d}z}{\mathrm{d}r} + \left(2Ar + \frac{A\varepsilon}{r^2}\right)z = rk(B_0 + \varepsilon B_1) \,. \tag{10}$$

Equation (10) can now be solved using the usual method of integrating factors, where we write the resulting integration constant as $C_0 + \varepsilon C_1$. Taylor-expanding wherever ε appears, and replacing for ε back in terms of g after the expansions, solving (10) gives

$$\mathrm{e}^{\frac{\nu}{2}} = z = z_0 + z_1 \,, \tag{11}$$

where z_0 corresponds to the well-known single-layer case, and z_1 is given by

$$z_1 = \frac{kgb}{2A} + \frac{kgB}{2}\left(\frac{1}{2r} - r\right) - \frac{gD}{2r\sqrt{1-Ar^2}} + gE\sqrt{Ar^2-1} \tag{12}$$

with

$$\mathrm{e}^{\lambda}_{(1)} = -\frac{g}{r}\,, \tag{13}$$

$$p_{(1)} = -\mu + bz_1\,, \tag{14}$$

where k, A, B, and D are zero-order constants. E, b, are first order.

The corresponding solutions of (2)–(5) for the dependent variables of the inner core of mass density ρ are of a similar form to the single-layer case. However, we remember that the slightly higher mass density of the inner core compared to the outer layer means that the relevant constants can be written as a perturbation of the single-layer case as follows:

$$\mathrm{e}^{\lambda}_{(\mathrm{c})} = 1-(A+\alpha)r^2\,, \tag{15}$$

$$\mathrm{e}^{\frac{\nu}{2}}_{(\mathrm{c})} = \frac{k(B+\beta)}{2(A+\alpha)} + (D+\delta)\sqrt{1-(A+\alpha)r^2}\,, \tag{16}$$

$$p_{(\mathrm{c})} = -\frac{3(A+\alpha)}{k} + (B+\beta)\,\mathrm{e}^{\frac{\nu}{2}}{}_{(\mathrm{c})}\,, \tag{17}$$

where the suffix (c) denotes core, and we have assumed that $\rho = \mu + \gamma$ and $A = \frac{1}{3}k\mu$. The next step is to Taylor-expand the perturbed quantities around the single-layer case (it may help to use a book-keeping parameter to keep track of the expansion orders). The resulting first-order expressions are of the form

$$\mathrm{e}^{\lambda}_{(\mathrm{c})(1)} = -\alpha r^2\,, \tag{18}$$

$$\mathrm{e}^{\frac{\nu}{2}}_{(\mathrm{c})(1)} = \alpha f_\alpha(r,A,B,D) + \beta f_\beta(r,A,B,D) + \delta f_\delta(r,A,B,D)\,, \tag{19}$$

$$p_{(\mathrm{c})(1)} = \alpha h_\alpha(r,A,B,D) + \beta h_\beta(r,A,B,D) + \delta h_\delta(r,A,B,D)\,, \tag{20}$$

where the lengthy functions f_α, f_β, f_γ, and h_α, h_β, h_γ depend on the independent variable r, and the single-layer constants A, B, D. One can immediately see that (18)–(20) are a linear system of algebraic equations in β and γ, and similarly for the outer layer a set of linear equations in E and B is formed from the $O(g)$ limits of (3), (6), and (11).

For the vacuum exterior

$$\mathrm{e}^{\lambda} = \frac{1}{1-\frac{2(M+\Delta M)}{r}}\,, \tag{21}$$

$$\mathrm{e}^{\frac{\nu}{2}} = \sqrt{1-\frac{2(M+\Delta M)}{r}}\,, \tag{22}$$

where M is the asymptotically measured mass of the single-layer case (*i.e.* the whole composite structure having a single mass density μ throughout), and ΔM is the asymptotically measured mass corresponding to the inner core that is over and above what would be provided by the single-layer case (*i.e.* ΔM originates from the perturbed portion of the inner core with mass density ρ–μ); both (21) and (22) must be Taylor-expanded to zero and first order in ΔM. The Darmois matching now takes place in two stages as follows:

(i) Zero-order: This is trivial, and is just the well-known single-layer case with mass density μ.

(ii) First-order: Once the above has been satisfied, the expansion of (6) to first order in g is matched to a first order expansion of (21) in ΔM, in order to obtain ΔM in terms of g. Similarly to first order in g, (11) is matched to (22), and (3) is matched to its vacuum value (*i.e.* zero); this allows us to simultaneously solve for E and b. Then to solve for the constants β and δ in an analogous manner, the parts of (3) and (11) that are of first order in g are respectively matched to (20) and (19). Matching the first-order part of (6) to (18) determines g in terms of α, in other words in terms of (ρ–μ), the perturbed mass density of the core.

One possible interpretation of the result obtained from this section is that the zero-order case corresponds to the self-gravity of the entire composite body with a single mass density μ, whereas the first-order result corresponds to a field in the outer layer being bathed in incompressible fluid, but itself not contributing any additional gravitating mass from the outer layer. This field would originate from the perturbed mass density of the core.

4. Large variation in the mass densities

Returning to (9), if we instead make the assumption that terms involving the quantity A are much smaller than terms involving g, then this implies that the equivalent single-layer total mass of the outer layer from $r = 0$ to r is much smaller than the total mass of the core. This can be achieved if the mass density of the outer layer is small compared to that of the core; it must be understood that unlike in the previous section, this approximation is *not* a perturbation of the single-layer case, and must be solved from first principles. Letting $\varepsilon = \frac{A}{g}$ and $B \to B_0 + \varepsilon B_1$, Eq. (9) can be written as

$$2\left(1 - \varepsilon g r^2 + \frac{g}{r}\right)\frac{\mathrm{d}z}{\mathrm{d}r} + \left(2\varepsilon g r + \frac{g}{r^2}\right) z = rk(B_0 + \varepsilon B_1)\,. \tag{23}$$

Equation (23) can now be solved using the usual method of integrating factors, where we write the resulting integration constant as $C_0 + \varepsilon C_1$.

Taylor-expanding wherever ε appears, and replacing for ε back in terms of $\frac{A}{g}$ after the expansions, solving (23) gives the quite lengthy expression

$$\mathrm{e}^{\frac{\nu}{2}} = z = z_1 + z_2 + z_3 + z_4\,, \tag{24}$$

where

$$\begin{aligned}
z_1 &= \left(2r^2 - 5gr - 15g^2\right)\frac{kB_0}{8}\left(1 - \frac{Ar^2}{2\left(1+\frac{g}{r}\right)}\right) \\
&\quad - \left(\frac{AkB_1}{256g}\right)\frac{\left(480g^3 + 640g^2r + 96gr^2 - 64r^3\right)}{(r+g)} \\
z_2 &= \sqrt{\left(1+\frac{g}{r}\right)}\left(16C_0 - \frac{15}{2}g^2kB_0\ln X\right)\left(1 - \frac{Ar^2}{32\left(1+\frac{g}{r}\right)}\right)\,, \\
z_3 &= -\frac{AkB_0}{256(r+g)}\left(3465g^5 + 4620g^4 + 693g^2r^2 - 198g^2r^3 + 88gr^4 - 48r^5\right)\,, \\
z_4 &= \frac{A}{g}\sqrt{1+\frac{g}{r}}\left(C_1 - \frac{3465}{1024}g^2k\left(g^3B_0 + \frac{32}{231}B_1\right)\ln X\right)\,, \\
X &= \frac{2\sqrt{r(r+g)} - 2r - g}{2\sqrt{r(r+g)} + 2r + g}\,,
\end{aligned}$$

and A is given by (7). As before, the corresponding solutions of (2)–(5) for the dependent variables of the inner core of mass density ρ are of a similar form to the single-layer case. However, as the mass density of the inner core is much greater than in the outer layer, this means that the relevant constants for the core must be written in terms of their counterparts in the outer layer as follows:

$$\mathrm{e}^{\lambda}_{(\mathrm{c})} = 1 - (\alpha + A)r^2\,, \tag{25}$$

$$\mathrm{e}^{\frac{\nu}{2}}_{(\mathrm{c})} = \frac{k(\beta + B)}{2(\alpha + A)} + (\delta + D)\sqrt{1 - (\alpha + A)r^2}\,, \tag{26}$$

$$p_{(\mathrm{c})} = -\frac{3(\alpha + A)}{k} + (\beta + B)\mathrm{e}^{\frac{\nu}{2}}{}_{(\mathrm{c})}\,, \tag{27}$$

where $\alpha = \frac{1}{3}k\gamma$ and, just like in the previous section, we have assumed that $\rho = \gamma + \mu$, but this time, μ is the small perturbation to the mass density. In other words, the effective mass density corresponding to this zero-order case is $\gamma = \rho - \mu$, and the zero-order quantities α, β, and δ are associated with γ (and not ρ); the first-order constants A, D, and B are now associated with μ. The next step is to Taylor-expand the perturbed quantities around the zero-order case (it may help to use a book-keeping parameter to keep track of the expansion orders). The resulting zero-order expressions are

$$\mathrm{e}^{\lambda}_{(\mathrm{c})(0)} = 1 - \alpha r^2 \,, \tag{28}$$

$$\mathrm{e}^{\frac{\nu}{2}}_{(\mathrm{c})(0)} = \frac{k\beta}{2\alpha} + \delta\sqrt{1-\alpha r^2} \,, \tag{29}$$

$$p_{(\mathrm{c})(0)} = -\frac{3\alpha}{k} + \beta\, \mathrm{e}^{-\frac{\nu}{2}}{}_{(\mathrm{c})} \,, \tag{30}$$

and the corresponding first order expressions are of the form

$$\mathrm{e}^{\lambda}_{(\mathrm{c})(1)} = -Ar^2 \,, \tag{31}$$

$$\mathrm{e}^{\frac{\nu}{2}}_{(\mathrm{c})(1)} = Af_A(r,\alpha,\beta,\delta) + Bf_B(r,\alpha,\beta,\delta) + Df_D(r,\alpha,\beta,\delta) \,, \tag{32}$$

$$p_{(\mathrm{c})(1)} = Ah_A(r,\alpha,\beta,\delta) + Bh_B(r,\alpha,\beta,\delta) + Dh_D(r,\alpha,\beta,\delta) \,, \tag{33}$$

where the lengthy functions f_A, f_B, f_D and h_A, h_B, h_D depend on the independent variable r, and the zero-order constants α, β, δ. One can immediately see that (28)–(30) is a system of equations in β and δ, and (31)–(33) is a linear system in B and D.

For the outer layer, a set of linear equations in B_0 and C_0 is formed from the $A = 0$ limits of (24), and a first-order set in B_1 and C_1 is formed from the $O(A)$ limit of (24).

As in the previous section, the vacuum exterior is given by

$$\mathrm{e}^{\lambda} = \frac{1}{1 - \frac{2(M+\Delta M)}{r}} \,, \tag{34}$$

$$\mathrm{e}^{\frac{\nu}{2}} = \sqrt{1 - \frac{2(M+\Delta M)}{r}} \,. \tag{35}$$

However, now M is the asymptotically measured mass of the inner core, and ΔM is the asymptotically measured mass corresponding to the (perturbed) self-gravity in the outer layer; this interpretation of the gravitating mass is somewhat different to the case of small mass density variation. Both (34) and (35) must be Taylor-expanded to zero and first order in ΔM, respectively, and respecting both orders of the approximation, the Darmois matching must take place in two distinct stages as follows:

(i) Zero-order: At the interface, Eq. (28) and the $A = 0$ limit of (6) are matched to express g in terms of α. Then β and δ can be found from equating (29) to the $A = 0$ limit of (24), along with the matching of pressure at the interface using both (30) and the zero-order limit of (3); re-arranging this latter equation makes it linear in β and δ. At

the outer boundary, the $A = 0$ limit of (6) is matched to the $\Delta M = 0$ limit of (34) in order to express M in terms of g, while B_0 and C_0 are obtained from matching the $A = 0$ limit of (24) with the $\Delta M = 0$ limit of (35), and also from matching the $A = 0$ limit of (3) to the vacuum.

(ii) First-order: Once the above has been satisfied, at the outer boundary, the perturbed $A \neq 0$ part of (6) is matched to the first-order expansion of (34) in ΔM, to obtain ΔM in terms of A. Then to solve for the constants B_1 and C_1, the parts of (24) that are of first order in A are matched to the parts of (35) that are of first order in ΔM, along with matching the perturbed $A \neq 0$ part of (3) to its vacuum value (in this case zero). At the interface (31) and the perturbed $A \neq 0$ part of (6) is trivially matched but to solve for B and D in terms of both B_1 and C_1, (32) is matched to the $A \neq 0$ part of (24), and (33) is matched to the perturbed $A \neq 0$ part of (3).

In a somewhat analogous manner to the case of small mass density variation, one could regard (24) as consisting of an 'exact' part with $A = 0$, supplemented by a perturbation with $A \neq 0$. The exact part corresponds to a field that is effectively bathed in the incompressible fluid of the outer layer, while not contributing any additional gravitating mass from the outer layer, and originates from the inner core. On the other hand, the perturbation corresponds to the self-gravity contributed by the outer layer.

5. Discussion

Now that we have obtained our static initial configuration, we ask ourselves how the collapse may proceed. Naively, one could try to apply the O–S model separately to the two layers, and then apply the Darmois conditions at the mutual interface (*e.g.* [11]). But it can easily be shown that this results in requiring the scale factors of both layers to be equal, something that is not compatible with the initial conditions of differing mass densities in both layers ([7] — see (11.9.23) and (11.9.25)).

Indeed, on physical grounds, one should expect at least one of the layers to have a scale factor that is *not* spatially uniform. For if both scale factors were different yet spatially independent, the outer layer would collapse at a slower rate than if both layers have the same mass density. If the outer layer fails to keep up with the inner layer, then a gap would open up at the mutual interface. In order to preserve continuity at the interface, the scale factor of the outer layer must now have a spatial dependence, in addition to its usual time dependence.

Therefore, the O–S model in its original form cannot be used for the outer layer, and it will have to be solved from first principles. But in the case of the small mass density variation, it may be possible to treat the outer layer as a perturbation of the single-layer O–S collapse, in an analogous manner to what we have done for the static case. For the large mass density variation, while the core can be modelled using the O–S description, the collapsing outer layer really would have to be solved from scratch, also using a perturbation decomposition that is analogous to what we have done for the corresponding static case.

Given the nature of our approximations that will be used in the spherical collapse of our two-layer dust ball, it is quite possible that at worst, the small mass density variation case will only result in a correspondingly small delay to the formation of the event horizon when compared against the single-layer O–S model, but the large mass density variations may well lead to a significant delay in horizon formation, or even worse a naked singularity.

One can then go on to ask what the effects of rotation may add. The PCM framework for non-spherical collapsing (single-layer) dust has the property that depending on the initial stationary rotation rate, the collapse can 'spin-up' the body such that the extreme Kerr limit is eventually reached, and a naked singularity results. It is not unreasonable to expect that, when the PCM framework is applied to the two-layer dust collapse, then this can combine with the above-mentioned effects from the spherical case in a quite non-trivial manner and significantly increase the risk that a naked singularity arises, even for relatively small variations in the mass density, and this must be investigated further.

6. Summary and next steps

In this study, we have introduced the concept of a two-layer homogeneous isotropic body, and solved for the equilibrium spherical case where the inner core is denser than the outer layer. In order to solve the equations, two separate approximations had to be used — both small and large mass density variations. As this solution constitutes initial data for spherical collapse, similar approximations will also have to be used during the collapsing stage.

Once the results for spherical collapse have been obtained, departures from spherical symmetry will be investigated using an analogous framework to Price–Cunningham–Moncrief, again for small mass density variations and separately for large mass density variations. This will provide an important clue on the nature of the end states when a slowly rotating body with multiple layers experiences a full collapse; and this imprint will manifest itself in the emitted gravitational waves.

Of course, gravitational collapse is not the only application for the results presented here; they can also be used as the basis to study other types of perturbations of multi-layer stars. Indeed, a parallel project currently being undertaken by the author is to investigate how gravitational waves emitted by non-spherical interface oscillations end up affecting the rotation of the star itself, and compare with the many previous investigations regarding the corresponding behavior generated by other types of oscillations [12].

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