EXAMINING QUASINORMAL MODE INSTABILITY WITH THE PSEUDOSPECTRUM*

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The gravitational waves emitted by compact objects such as neutron stars or black holes are characterized by quasinormal modes (QNMs). The QNM frequencies encode information about the relic object, and — in the most energetic cases — are sensitive to higher-curvature corrections to General Relativity. Their stability depends on how the frequencies change in response to perturbations, and the pseudospectrum allows us to quantify these changes. In this work, we review how the pseudospectrum is used to examine the stability of the gravitational modes of Schwarzschild black holes. We then discuss ongoing work into applying these methods to the quasinormal modes of a Yang–Mills soliton.

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1. Introduction

Black hole quasinormal modes (QNMs) encode the resonant response to linear perturbations of the spacetime and their frequencies give us information about the characteristics of the black hole. For instance, the fundamental frequency of a spherically-symmetric Schwarzschild black hole in (3+1) dimensions depends on the black hole mass [1]. In the era of gravitational wave spectroscopy, we can use quasinormal modes to determine characteristics of astrophysical black holes.

The analytical study of quasinormal modes of black holes began long before the first operational gravitational interferometer, most notably with works by Leaver [2], Nollert [3], and others [4]. It was noticed immediately that these spectra were not always stable — in fact, different overtones had different stability characteristics [5]. How the overtones react to different perturbations is significant beyond the question of the stability of the

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spectrum itself. Since overtones are able to probe the region close to the event horizon, they offer a window into a regime where deviations from General Relativity could be detected in astrophysical black holes [6, 7].

The pseudospectrum provides the mathematical structure required to systematically study the stability of the spectrum of quasinormal modes. Calculating the pseudospectrum of an operator gives a type of topographic map of quasinormal mode instability, where contour lines indicate the direction of flow of unstable modes and the magnitude provides a measure of the size of perturbation required to shift a quasinormal mode. However, the construction of the pseudospectrum relies on a more rigorous definition of QNMs than is normally employed and this is achieved via the process of hyperboloidal compactification. Using hyperboloidal slices to approach future null infinity allows us to define the spectral problem in terms of singular Sturm–Liouville operators, thereby replacing the imposition of boundary conditions on the wave function by the requirement of regularity of the eigenfunctions of the operator.

This work is organized into two main parts. First, in Section 2, we discuss the application of hyperbolic compactifications, define the pseudospectrum, and review results from [8] regarding the stability of the QNMs of spherically-symmetric black holes. Then in Section 3, we introduce solitons, their dynamics, and show how the pseudospectrum can be used to investigate signs of resonant behaviour in their final states. We also present in-progress findings and discuss ongoing research.

2. Quasinormal modes and the pseudospectrum

Without loss of generality, consider a massless scalar field on a stationary, spherically-symmetric, black hole background. In the standard Schwarzschild coordinates, we have

$$\Box\mathit{\Psi} = 0\,, \qquad \mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right)\,. \tag{2.1}$$

After rescaling the field by $\mathit{\Psi} = r^{-1}\phi$, we can define the tortoise coordinate generically as $\mathrm{d}r/\mathrm{d}r_* = f(r)$. Spherical symmetry and time independence of the metric functions imply a decomposition of the scalar field such that we recover the Schrödinger-like master equation[1]

$$\frac{\partial^2\phi}{\partial r_*^2} = \left(V - \omega^2\right)\phi\,. \tag{2.2}$$

[1] For the sake of simplicity, radial and angular indices are suppressed throughout the discussion, and coordinates are defined only up to a rescaling factor. See [8] for details.

In these coordinates, the black hole horizon is located at $r_* \to -\infty$ while spatial infinity is $r_* \to \infty$, and the potential term goes to zero at each limit. The conventional asymptotic form of ϕ is taken to be a superposition of free waves. Since classical solutions can neither emerge from the event horizon nor enter the system from infinity, the solutions for ϕ at these limits are normally restricted to be either purely incoming or purely outgoing waves. However, unlike a purely spatial problem, the spacetime definition of the radiative zone (where degrees of freedom propagate as free waves) is not simply $r_* \to \infty$. Rather, it is future null infinity $\mathscr{I}^+$, defined as the limit of $r_* \to \infty$ at constant retarded time $u = t - r_*$ that defines the radiative zone. A hyperboloidal approach to $\mathscr{I}^+$ benefits from allowed coordinate redefinitions while also respecting the radiative limit.

Since energy can leave the system through either boundary, the system is non-conservative. The evolution of any non-conservative system is described by a non-self-adjoint (non-Hermitian) operator with quasinormal modes as its eigenvalues [9]. This definition informs the procedure for calculating the QNMs: define an approach to $\mathscr{I}^+$ in terms of a spacelike hypersurface, then recast the QNM problem as an eigenvalue problem for a non-self-adjoint operator.

2.1. Hyperboloidal compactification

We now apply the unspecified hyperboloidal compactification

$$\begin{cases} t = u - h(x) \\ r_* = g(x) \end{cases} \tag{2.3}$$

to (2.2), where $u =$ const. is a horizon-penetrating hyperboloidal slice that intersects $\mathscr{I}^+$. Under this transformation, the scalar field master equation becomes

$$\left(\left[1-\left(\frac{h'}{g'}\right)^2\right]\partial_u^2 - \frac{2}{g'}\left[\frac{h'}{g'}\right]\partial_{ux}^2 - \frac{1}{g'}\partial_x\left[\frac{h'}{g'}\right]\partial_u - \frac{1}{g'}\partial_x\left[\frac{1}{g'}\partial_x\right] + V\right)\phi = 0\,, \tag{2.4}$$

Performing a time reduction via $\psi \equiv \partial_u \phi$, we can write (2.4) in terms of two Sturm–Liouville operators, L_1 and L_2, acting on ϕ and ψ

$$\partial_u^2 \phi = L_2 \partial_u \phi + L_1 \phi\,, \tag{2.5}$$

with the operators defined as

$$L_1 = \frac{1}{\rho(x)}\left(-\partial_x(p(x)\partial_x) + \hat{V}(x)\right)\,, \tag{2.6}$$

$$L_2 = \frac{1}{\rho(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)\,. \tag{2.7}$$

Note that $p(x)$ must satisfy $p(x = \pm 1) = 0$ and $\rho(x)$ must be strictly positive on the open interval $x \in (-1, 1)$ [10]. These characteristics, plus regularization conditions on the eigenfunctions, encode the classically-motivated boundary conditions into the construction of the operator L_1.

Finally, we define the vector $\varPhi = (\phi, \psi)^T$ and use a Fourier decomposition of the form $\varPhi(u, x) = \varPhi(x)\,\mathrm{e}^{i\omega u}$ such that the spectral problem is

$$\begin{pmatrix} 0 & 1 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = i\omega \begin{pmatrix} \phi \\ \psi \end{pmatrix} . \tag{2.8}$$

2.2. The pseudospectrum

To introduce the pseudospectrum, consider an operator A with left- and right-hand eigenvectors w_i and v_i that solve

$$A^\dagger w_i = \bar{\lambda}_i w_i \qquad \text{and} \qquad A v_i = \lambda_i v_i \,. \tag{2.9}$$

The perturbed operator $A(\epsilon) = A + \epsilon\,\delta A$ with $\epsilon > 0$ and $||\delta A|| = 1$ can similarly be solved by eigenvectors $w_i(\epsilon)$, $v_i(\epsilon)$ whose eigenvalues are $\bar{\lambda}_i(\epsilon)$, $\lambda_i(\epsilon)$. The perturbed eigenvalues satisfy

$$|\lambda_i(\epsilon) - \lambda_i| \leq \epsilon \frac{||w_i||\,||v_i||}{|\langle w_i, v_i\rangle|} + \mathcal{O}\left(\epsilon^2\right) . \tag{2.10}$$

When A is a self-adjoint operator, $A = A^\dagger$ and the difference between perturbed and unperturbed eigenvalues is of the order of ϵ. This is referred to as spectral stability: a perturbation of the operator produces a shift in eigenvalues that is proportional to the size of the perturbation. On the other hand, when the operator is non-self-adjoint, it may be the case that $|\langle w_i, v_i\rangle| \ll 1$ and so the shifted eigenvalues are much farther from the unshifted spectrum. This is known as spectral instability.

The pseudospectrum quantifies the stability of an operator by comparing the deviation of the perturbed spectrum from the unperturbed one through the resolvent of the operator $R_A(\lambda) = (\lambda\mathbb{1} - A)^{-1}$. This definition is more conducive to use with non-self-adjoint operator, and is given by

$$\sigma^\epsilon(A) = \left\{ \lambda \in \mathbb{C} : ||R_A(\lambda)|| = ||(\lambda\mathbb{1} - A)^{-1}|| > \frac{1}{\epsilon} \right\} . \tag{2.11}$$

Note that the discussion of spectral stability and the pseudospectrum always relies on an operator norm to set the scale of the deviations from the unperturbed spectra. Therefore, it is essential to define the scalar inner product that determines the norm. Since the system is non-conservative, the conventional definition of the norm is no longer appropriate. Instead, the total energy contained in a spatial slice and associated with the scalar field in (2.4) provides a more natural basis [11]. In terms of the time-reduced variables, the energy norm for solutions to the spectral problem of Eq. (2.8) is defined as

$$||\Phi||_E^2 = E(\phi, \psi) = \frac{1}{2} \int_a^b \left(\rho(x) |\psi|^2 + p(x) |\partial_x \phi|^2 + \hat{V}(x) |\phi|^2 \right) \mathrm{d}x \,. \tag{2.12}$$

Using the definition $||\Phi||_E^2 = \langle \Phi, \Phi \rangle_E$, the energy inner product is

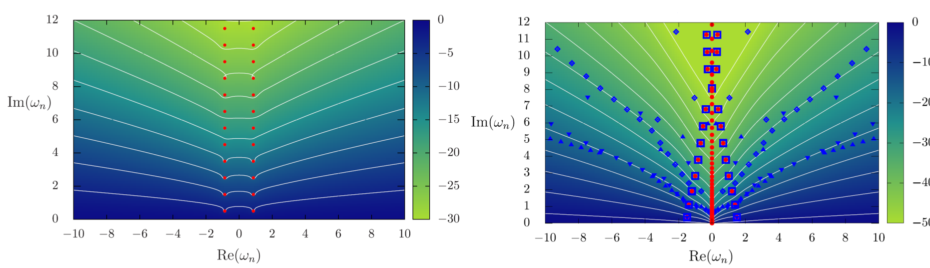
$$\langle \Phi_1, \Phi_2 \rangle = \frac{1}{2} \int_a^b \left(\rho(x) \bar{\psi}_1 \psi_2 + p(x) \partial_x \bar{\phi}_1 \partial_x \phi_2 + \hat{V} \bar{\phi}_1 \phi_2 \right) \,. \tag{2.13}$$

When constructing the pseudospectrum for the system (2.8), we must use the energy norm $|| \cdot ||_E$.

We have now established the essential ingredients for the study of quasinormal modes: *(i)* the hyperboloidal compactification, *(ii)* defining a non-self-adjoint operator whose eigenvalues are the QNM, and *(iii)* the energy inner product.

2.3. Schwarzschild black hole

In [8], the pseudospectra for different scalar field modes on a Schwarzschild background[2] were calculated (see figure 1). First, the Pöschl–Teller potential was used to approximate the potential term for a purely radial scalar mode, *i.e.* when $\ell = 0$. In figure 1 (a), the contour lines predict



(a) $\ell = 0$ modes (b) $\ell = 2$ modes

Fig. 1. Pseudospectra for scalar field modes around a Schwarzschild black hole in [8]. Eigenvalues are indicated by red circles; $\log_{10} \epsilon$ is indicated by the colour bar on the right.

[2] See [12] for the Riessner–Nordström geometries.

that the fundamental mode ω_0 is the most stable, while higher overtones are increasingly unstable. In figure 1 (b), the pseudospectrum of the $\ell = 2$ scalar modes is shown, overlaid with the response to an added perturbation of $\delta\hat{V}(x) = \cos(2\pi kx)$ to the rescaled potential $\hat{V}(x)$ in (2.6). Varying the magnitude and frequency of $\delta\hat{V}(x)$ produces eigenvalues shown as blue markers. Note the agreement with the contour lines of the pseudospectrum.

Given how much information the pseudospectrum can provide about the stability of the QNMs, we now wish to apply this method to a new type of system: the soliton.

3. Solitons

Solitons are broadly defined as localized solutions that carry topological charge. Solitons that interpolate between degenerate minima are known as *kinks* and are some of the simplest examples of solitons, being present in (1+1)-dimensional field theories such as the sine-Gordon model and ϕ^4 theory. Despite their simple structure, the dynamics of kink collisions are governed by resonant scattering due to the transfer of energy between kinematic and internal degrees of freedom [13]. In some cases, resonant scattering is mediated by the quasinormal modes of the kinks [14].

Asymptotically, solitons have been proven to behave in a much more predicable manner: the final state must always be a simple superposition of a static soliton and an alternating series of rescaled solitons, plus radiation [15]. In [16], we demonstrated this conjecture for the (4+1)-dimensional, equivariant Yang–Mills soliton. We also noted a bifurcation in the final state depending non-monotonically on a single input parameter. In order to determine if this is a result of a resonant process mediated by QNMs, we will examine the stability of the kink spectrum using the pseudospectrum.

Consider the equation for the Yang–Mills potential $f(t,r)$ in 4+1 dimensions with $r \in [1,\infty)$

$$\frac{\partial^2}{\partial t^2} f(t,r) = \frac{\partial^2}{\partial r^2} f(t,r) + \frac{2f(t,r)}{r^2}\left(1 - f^2(t,r)\right) . \tag{3.1}$$

Equation (3.1) possesses two vacua, namely $f = \pm 1$. A static solution known as the *half-kink*, $Q(r)$, is a global minimizer of energy. To examine the decay of perturbations around the half-kink, we defined the null coordinate $u = t - r$ and compactified the space via $x = r^{-1/2}$. After linearizing around the half-kink, we found a pair of quasinormal modes[3] $\omega = \pm 0.476858 - 0.364322i$ analytically using Leaver's method [2] and verified the result numerically.

[3] In [16] the convention for QNM frequencies *did not* include an explicit factor of i. Here we rewrite the result to agree with the convention of this work.

3.1. Hyperboloidal compactification and the Yang–Mills soliton

Following the prescription for hyperboloidal compactification in [7], we can rewrite the equation for f in terms of unspecified compactification functions $g(x)$ and $h(x)$. Linearizing around the half-kink, $f(u,x) = Q(x) + c(x)\phi(u,x)$, where $c(x)$ is a rescaling function that is used to write the equation for ϕ in terms of the Sturm–Liouville operators as in (2.8). In figure 2, we show the result of calculating the eigenvalues of perturbations around the half-kink at increasing resolutions. We recover one complex conjugate pair of eigenvalues and three small, purely imaginary eigenvalues. The purely imaginary eigenvalues correspond to non-oscillating decays and, due to their small coefficients, will form the asymptotic tails observed in [16].

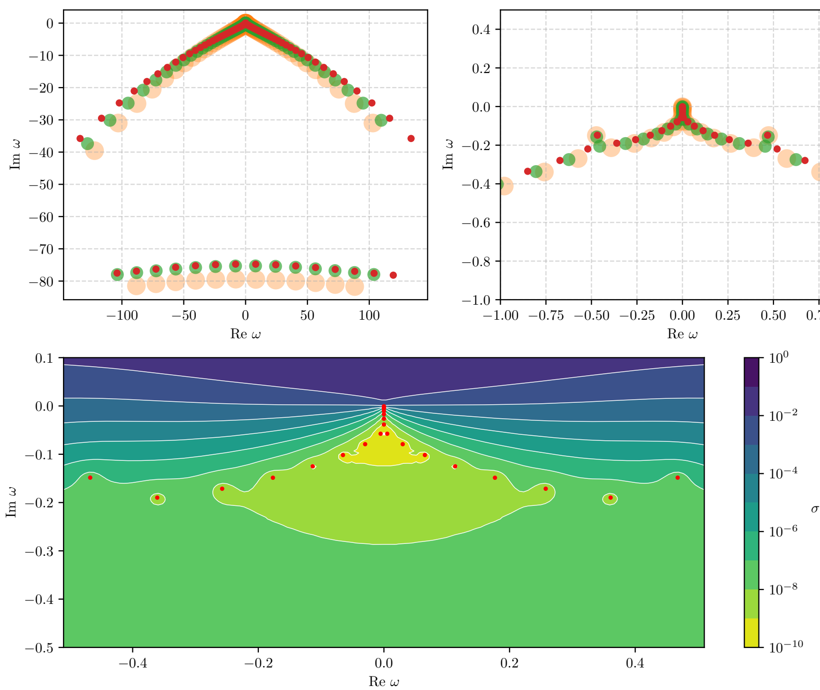


Fig. 2. Top: The eigenvalues for linear perturbations around the half-kink for resolutions $N = 100$ (large orange circles), $N = 110$ (medium green circles), and $N = 120$ (small red circles). Bottom: the pseudospectrum with eigenvalues overlaid.

As work on the application of the pseudospectrum to solitons continues, we will examine the response of the eigenvalues to both random and deterministic perturbations of the potential term in (2.6). Of particular interest will be any stability of the spectrum to specific frequencies, as stable QNMs may facilitate the resonant behaviour responsible for the observed bifurcation in asymptotic states.

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