ANALYSIS OF TOPOLOGICALLY NON-TRIVIAL SOLUTIONS OF MAXWELL EQUATIONS IN DE SITTER SPACETIME*

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Construction of a generalization of electromagnetic Hopfions for de Sitter spacetime is briefly presented. We analyze non-trivial properties of field lines of the solution.

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1. Introduction

In Minkowski spacetime, Maxwell's equations allow for curious solutions characterized by non-trivial topological properties of field lines. One interesting example is the Hopfion solution. The Hopfion is a 'solitonary' solution of Maxwell[1] theory which has rich a topological structure related to the Hopf fibration. The characteristic structure of Hopfion can be easily seen on the integration curves of the vector field (see [2]). The structure of closed, linked field lines of Hopfions propagates without intersections along the light cone.

The aim of the paper is to obtain a Maxwell field on de Sitter background with analogical properties. The results are organized as follows: In Section 2, a brief survey of used reduced electromagnetic data is presented, together with a construction of a family of Hopfion-like solutions in de Sitter spacetime. Section 3 contains an analysis of topological properties of the generalized Hopfion in de Sitter spacetime. Analogically, Hopfion-like solutions in de Sitter spacetime can be obtained for linearized gravity. It is briefly described in Section 4.

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[1] In general, the Hopfion solutions of spin-N field are known in Minkowski spacetime.

1.1. Conformal flatness of de Sitter spacetime

Let us recall the standard form of de Sitter metric in stationary coordinates

$$g = -f(r)^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)^2} + r^2 \sigma_{AB} \mathrm{d}x^A \, \mathrm{d}x^B \,, \tag{1}$$

where $\sigma_{AB}\, \mathrm{d}x^A \, \mathrm{d}x^B = \mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2$ is a standard unit sphere metric and x^A denotes angular coordinate. We have $f(r) = \sqrt{1-k^2r^2}\,, k \in \mathbb{R}_+$, $r \in \mathbb{R}_+ \setminus \{\frac{1}{k}\}$, and θ, ϕ being the standard coordinates parameterizing a two-dimensional sphere. We ignore the coordinate singularities $\sin\theta = 0$. The metric can be transformed into the conformally Minkowskian form

$$g_{\mathrm{dS}} = \frac{4}{(1+k^2S^2)^2} \left(-\mathrm{d}T^2 + \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2 \right) \,, \tag{2}$$

where $S^2 = -T^2 + X^2 + Y^2 + Z^2$. The coordinate transformation which brings (2) into (1) reads

$$\begin{cases} T &= \frac{f(r)\sinh kt}{k(1-f(r)\cosh kt)}\,, \\ X &= \frac{r\sin\theta\cos\phi}{1-f(r)\cosh kt}\,, \\ Y &= \frac{r\sin\theta\sin\phi}{1-f(r)\cosh kt}\,, \\ Z &= \frac{r\cos\theta}{1-f(r)\cosh kt}\,. \end{cases} \tag{3}$$

2. Reduced data for Hopfions-like solutions in de Sitter spacetime

2.1. Brief overview of reduced data description

In the paper, we use the reduced data which in its principle is based on the conformal Yano–Killing two-form[2]. In particular, it is defined for type-D spacetimes. The construction for Kerr spacetime is presented in [4].

The proposed reduced data has turned out to be very effective in the analysis of Hopfions in Minkowski spacetime [5]. In the case of de Sitter spacetime, we perform our research in static coordinates (t, r, θ, ϕ) in which the metric has the form of Eq. (1). Consider a foliation of two-dimensional

[2] The conformal Yano–Killing is a generalization of the conformal Killing covector field to anti-symmetric two-forms. The reduced data is geometrically given by

$$\Phi = \frac{\imath}{2} \left[F_{\mu\nu} - \imath(*F_{\mu\nu}) \right] Q^{\mu\nu} \,, \tag{4}$$

where $\imath^2 = -1$ and $F_{\mu\nu}$ and $*F_{\mu\nu}$ are Maxwell field and its Hodge dual companion respectively. $Q_{\mu\nu}$ denotes the conformal Yano–Kiling tensor. See [4] for details.

spheres parameterized by radius r. Quasi-locally, on each sphere, vacuum Maxwell equations in a complex form can be encoded in terms of reduced data

$$\begin{aligned} \partial_k\left(\sqrt{\det h}Z^k\right) &= 0\,, \\ \partial_t Z^k &= -\frac{\imath}{\sqrt{\det h}}\partial_l\left(N\epsilon^{klm}Z_m\right)\,, \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \varPhi &= \frac{rZ^r}{f(r)}\,, & (5)\\ \partial_r(r\varPhi) &= -\frac{r^2}{f(r)}Z^A_{||A}\,, & (6)\\ \partial_t(r\varPhi) &= \imath\frac{r^2}{\sqrt{\det h}}\epsilon^{AB}\partial_B Z_A\,, & (7)\end{aligned}$$

where (5) is the definition of reduced data. $Z = E + \imath B$ is called the Riemann–Silberstein vector field. Let Σ_t be a spatial part of $\{t = \mathrm{const.}\}$ hypersurface. Coordinates on Σ_t contain small Latin indices. The induced metric on Σ_t is denoted by $h_{kl}\mathrm{d}x^k\mathrm{d}x^l$. $Z^A_{||A} = 1/\sqrt{\det\sigma}\partial_A(\sqrt{\det\sigma}Z^A)$ is a two-dimensional divergence on a unit sphere. The Levi-Civita tensors obey $\epsilon^{r\theta\phi} = \epsilon^{\theta\phi} = 1$.

2.2. Generalization of reduced data for Hopfions

For the construction of the Hopfion-like solutions in de Sitter space, we use the formalism of reduced data. We highlight that simply conformally transforming the Hopfion electromagnetic tensor from Minkowski space to de Sitter would, in principle, result in a valid solution due to Maxwell equations being conformally covariant. However, such a transformation does not preserve the surfaces of constant time, thus resulting in a solution without the remarkable topological properties of the Hopfion field. Therefore, we construct the electromagnetic field *in situ*, using reduced data obtained as a conformal transformation of the Hopfion scalar for Minkowski space.

In Minkowski space, the reduced data for generalized Hopfions[3] — l-parameter family of electromagnetic solutions (see [5]) in Cartesian coordinates reads

$$\varPhi_l = \frac{(Z+\imath X)^l}{\left[-(T-\imath\beta)^2 + X^2 + Y^2 + Z^2\right]^{l+1}}\,. \tag{8}$$

Applying the conformal transformation (3) to Hopfion reduced data (8), we obtain, up to a constant overall factor, a formula for reduced data in static coordinates

[3] In [5], equation (8) is given, up to rotation, as (2.12) with $\beta = 1$. Let us highlight that rescaling of the coordinates $(t, x, y, z) \to (\alpha t, \alpha x, \alpha y, \alpha z)$ enables one to set $\beta = 1$ for non-zero β. During the rescaling, $\varPhi_l$ modifies up to a constant overall factor, which does not change the behavior of the solution.

$$\Phi_l = \frac{r^l(\cos\theta + \imath\sin\theta\cos\phi)^l}{[(1+k^2\beta^2) + (1-k^2\beta^2)\, f(r)\cosh kt + 2\imath k\beta f(r)\sinh kt]^{l+1}}\,. \tag{9}$$

However, the reduced data (9) do not lead to (8) as a limit $k \to 0$. It can be corrected by a rescaling of the β parameter as

$$\beta = \frac{2}{k^2\tilde{\beta}}\,, \tag{10}$$

then the reduced data reads

$$\Phi_l = \frac{r^l(\cos\theta + \imath\sin\theta\cos\phi)^l \left(k^2\tilde{\beta}^2\right)^{-l-1}}{\left[\left(\left(\tilde{\beta}^2k^2 - 4\right)\cosh(kt) + 4\imath\tilde{\beta}k\sinh(kt)\right) f(r) + k^2\tilde{\beta}^2 + 4\right]^{l+1}}\,. \tag{11}$$

Passing to the limit $k \to 0$, we obtain up to the overall constant factor equation (8) with $\beta = \tilde{\beta}$.

3. Topological properties of the solutions

Setting $l = 1$ and rescaling by a constant overall factor equation (11), we obtain the generalization of classical Hopfion for de Sitter spacetime. By Σ_0 we denote spatial part of $\{t = 0\}$. We wish to analyze topological properties of this particular choice of solutions on Σ_0 including the behavior near the horizon.

Let us display explicitly the generalization of classical Hopfion for de Sitter spacetime

$$Z^r = \frac{f(r)(\cos\theta + \imath\sin\theta\cos\phi)}{\Upsilon^2}\,, \tag{12}$$

$$Z^\theta = \frac{\xi\sin\theta + \imath\zeta\sin\phi - \imath\xi\cos\phi\cos\theta}{r^2}\,, \tag{13}$$

$$Z^\phi = \frac{\imath\zeta\cos\phi\cos\theta + \imath\xi\sin\phi - \zeta\sin\theta}{\sin\theta r^2}\,, \tag{14}$$

where

$$\Upsilon = \left(\tilde{\beta}^2k^2\cosh(kt) + 4\imath\tilde{\beta}k\sinh(kt) - 4\cosh(kt)\right) f(r) + k^2\tilde{\beta}^2 + 4\,, \tag{15}$$

$$\xi = -\frac{\left(\left(k^2\tilde{\beta}^2 - 4\right)\cosh(kt) + 4\imath\tilde{\beta}k\sinh(kt) + \left(k^2\tilde{\beta}^2 + 4\right) f(r)\right) r}{\Upsilon^3}\,, \tag{16}$$

$$\zeta = \frac{\imath r^2\left(k^3\tilde{\beta}^2\sinh(kt) + 4\imath\tilde{\beta}k^2\cosh(kt) - 4k\sinh(kt)\right)}{\Upsilon^3}\,. \tag{17}$$

The solution is explicitly obtained from (11) and (5)–(7). The recovery procedure of Z from Φ is based on the Hodge–Kodaira decomposition[4].

The generalized Hopfions in de Sitter spacetime share the remarkable topological properties of classical Hopfions in Minkowski spacetime (see [6]), in that the electric and magnetic field lines on Σ form toroidal structures analogical to Hopf fibration. Recall that the Hopfion-like solution is given in the static coordinates (t, r, θ, ϕ) of de Sitter spacetime. To investigate the field lines analytically, we introduce radial rescaling, given by

$$r = \frac{4R}{(4 + k^2 R^2)}\,, \tag{19}$$

where $R = \sqrt{x^2 + y^2 + z^2}$. With this transformation, the induced three-dimensional metric is expressed as

$$g|_{\{t=0\}} = \frac{16}{(4 + k^2 R^2)^2} \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,. \tag{20}$$

The associated Cartesian-like set of coordinates is related with toroidal coordinates by

$$x \;=\; \tilde{\beta}\frac{\sinh\eta}{\cosh\eta - \cos\sigma}\cos\phi\,, \tag{21}$$

$$y \;=\; \tilde{\beta}\frac{\sinh\eta}{\cosh\eta - \cos\sigma}\sin\phi\,, \tag{22}$$

$$z \;=\; \tilde{\beta}\frac{\sin\sigma}{\cosh\eta - \cos\sigma}\,, \tag{23}$$

and gives rise to a simple form of the electric field E^i on Σ_0:

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[4] Briefly, the angular part of the Riemann–Silberstein vector can be decomposed into gradient and co-gradient of some potentials. In terms of the co-vector field

$$Z_A = \partial_A\xi + \left(r^2\sin\theta\right)^{-1} g_{AC}\epsilon^{CB}\partial_B\eta\,. \tag{18}$$

Equations (5)–(7) enable one to find formulae for potentials. Defining $\widetilde{\Phi}_{l=1}$ by

$$\Phi_{l=1} = \widetilde{\Phi}_{l=1}(\cos\theta + \imath\sin\theta\cos\phi)\,,$$

where $\widetilde{\Phi}_{l=1}$ is a function of t, r only, we obtain

$$\xi = \frac{1}{2}\sqrt{1 - k^2 r^2}\partial_r\left(r\widetilde{\Phi}_{l=1}\right)\,, \qquad \zeta = \frac{i}{2\sqrt{1 - k^2 r^2}}\partial_t\left(r\widetilde{\Phi}_{l=1}\right)\,.$$

$$E^\eta = 0\,, \tag{24}$$

$$E^\sigma = -\frac{(\cosh\eta-\cos\sigma)\left(\left(4+k^2\tilde\beta^2\right)\cosh\eta-\left(4-k^2\tilde\beta^2\right)\cos\sigma\right)^2}{64\tilde\beta\cosh^3\eta}\,, \tag{25}$$

$$E^\phi = \frac{(\cosh\eta-\cos\sigma)\left(\left(4+k^2\tilde\beta^2\right)\cosh\eta-\left(4-k^2\tilde\beta^2\right)\cos\sigma\right)^2}{64\tilde\beta\cosh^3\eta}\,. \tag{26}$$

Given that the η component of the electric field is zero, the field lines are confined to two-dimensional surfaces parameterized by η, which turn out to be tori given by the family of equations in the Cartesian-like coordinates

$$z^2+\left(\sqrt{x^2+y^2}-\tilde\beta\coth\eta\right)^2=\frac{\tilde\beta^2}{\sinh^2\eta}\,. \tag{27}$$

Indeed, the structure of field lines, which is presented in figure 1, forms a structure of linked, closed circles. Each two lines are linked only once. Choosing the initial data properly on the tori-surface, the integral curves do not leave the tori. The field line on the symmetry axis is analytically investigated in Section 3.1. The magnetic field manifests a similar toroidal structure but rotated in the ZX plane by 90 degrees.

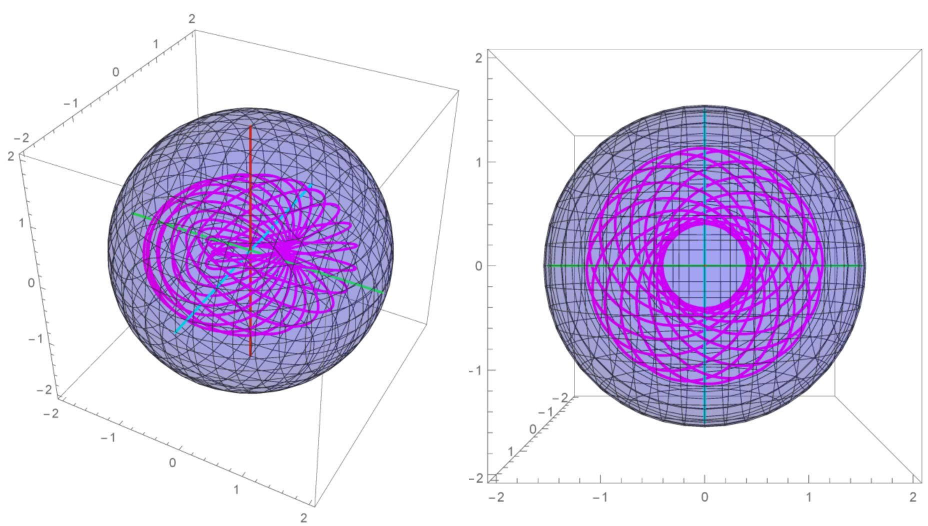

Fig. 1. Field lines for the electric field, forming toroidal structures inside the cosmological horizon. Both figures present a particular choice of integral curves in static coordinates from different perspectives. The sphere represents the cosmological horizon.

The most intriguing questions are raised for

$$\eta \leq \log\left(\frac{2+k\tilde{\beta}}{2-k\tilde{\beta}}\right), \tag{28}$$

for which the topological structure of generalized Hopfions is interacting with the cosmological horizon. One can check that for η fulfilling (28), the tori intersect non-trivially with the horizon. We plan to give a detailed analysis of the behavior of the field in a separate paper [1].

3.1. Field line which reaches the horizon in finite time

The structure of the electric field given by formulas (12) allows us to investigate two distinguished field lines on Σ_0, corresponding to two degenerate tori — the OZ axis and a ring in the XY plane. The electric field at $t = 0$ in static coordinates is explicitly given by

$$E^r = \frac{4k^4\tilde{\beta}^4\sqrt{1-k^2r^2}\cos\theta}{\left(k^2\tilde{\beta}^2+4+\left(k^2\tilde{\beta}^2-4\right)\sqrt{1-k^2r^2}\right)^2}, \tag{29}$$

$$E^\theta = -\frac{4k^4\tilde{\beta}^4\left(\left(-4+k^2\tilde{\beta}^2\right)+\left(4+k^2\tilde{\beta}^2\right)\sqrt{1-k^2r^2}\right)}{r\left(4+k^2\tilde{\beta}^2+\left(-4+k^2\tilde{\beta}^2\right)\sqrt{1-k^2r^2}\right)^3}\sin\theta, \tag{30}$$

$$E^\phi = \frac{16k^6\tilde{\beta}^5}{\left(4+\tilde{\beta}^2k^2+\left(-4+\tilde{\beta}^2k^2\right)\sqrt{1-k^2r^2}\right)^3}. \tag{31}$$

Thus, setting $\theta = 0$, finding the integral curve reduces to a one-dimensional problem. The equation

$$\dot{r}(s) = E^r(r(s)) \tag{32}$$

is readily solved by the separation of variables. A direct calculation shows that a field line starting at the origin reaches the horizon in finite time, precisely given by

$$s_{\text{horizon}} = \frac{(8+3\pi)k^4\tilde{\beta}^4+8\pi k^2\tilde{\beta}^2+16(3\pi-8)}{16k^5\tilde{\beta}^4}. \tag{33}$$

In turn, we can set $\theta = \frac{\pi}{2}$. The radial component vanishes, so as long as the θ-component of the field would remain zero, a circular integral line in the XY-plane would arise. Setting $E^\theta = 0$, we obtain an equation for the radius of the said field line

$$r_{\text{focus}} = \frac{4\tilde{\beta}}{4+k^2\tilde{\beta}^2}. \tag{34}$$

This field line in conformal coordinates coincides with the focal ring of the tori given by (27).

4. Linearized gravity Hopfion-like solutions on de Sitter background

The reduced data for electromagnetism, presented in Section 2.1, is generalized for linearized gravity. In the case of the Kottler background, the construction is given in [3]. A dedicated analysis for wave solutions on the de Sitter background is a generalization of appendix C.3 in [5]. It will be presented in [1]. We highlight that the reduced data

$$\Psi_l = \Phi_l \,, \tag{35}$$

viewed (11) as reduced data for the linearized gravity. The linearized gravity solution has analogical properties to the EM solution. It will be analyzed in detail in a separate paper [1].

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