QUANTUM CHAOS OF THE BELINSKI–KHALATNIKOV–LIFSHITZ SCENARIO*

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We quantize the solution to the massive model of the Belinski–Khalatnikov–Lifshitz (BKL) scenario using the integral quantization method. Classical deterministic chaotic behavior of the BKL scenario turns under quantization into stochastic chaos.

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1. Introduction

The Belinski, Khalatnikov, and Lifshitz (BKL) conjecture states that general relativity includes the solution with generic gravitational singularity [1, 2]. The evolution towards the BKL singularity, the so-called BKL scenario, consists of the deterministic dynamics turning into chaotic process near the generic singularity. The aim of this paper is the examination of the fate of the BKL chaos at quantum level.

The evolution process presented in [1, 2] is complicated and difficult to map into quantum evolution. There exists well defined and comparatively simple model of the BKL scenario [3–5] that can be used in the derivation of the BKL conjecture [6]. We call it the massive model of the BKL scenario. The model has been obtained from the general model of the Bianchi IX spacetime for perfect fluid under the assumption that in the dynamics, near the singularity, the anisotropy of space grows without bound so that each of the so-called directional scale factors oscillates, but never crosses each other, and evolves towards vanishing, *i.e.*, singularity. The resulting dynamics, specified in the next section, is different from the commonly known

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mixmaster dynamics [7, 8] consisting of infinitely many crosses among directional scale factors. The mixmaster model is the vacuum Bianchi IX model so that it is devoid of the details of the matter model of the BKL scenario. We have made a comparison of the dynamics of both models in [9]. The massive model of the BKL scenario is the subject of this article.

Recently, we have found that the classical dynamics underlying the present paper is generically unstable turning into a chaotic process near the singularity [10]. This feature is consistent with the original BKL scenario [1, 2].

Our quantization method, applied recently to the quantization of the Schwarzschild spacetime [11], includes quantization of the temporal and spatial variables on the same footing. The rationale for such an approach is that the distinction between time and space variables violates the general covariance of arbitrary transformations of temporal and spatial coordinates.

Quantization of the instability presented in [10] can be used in the examination of the fate of the chaos of the BKL scenario at quantum level [12].

2. Solution to the BKL scenario

To have the paper self-contained, we recall the main results of Ref. [10].

The dynamics of the massive model of the BKL scenario is defined as follows [3, 5]:

$$\frac{\mathrm{d}^2 \ln a}{\mathrm{d}t^2} = \frac{b}{a} - a^2\,, \qquad \frac{\mathrm{d}^2 \ln b}{\mathrm{d}t^2} = a^2 - \frac{b}{a} + \frac{c}{b}\,, \qquad \frac{\mathrm{d}^2 \ln c}{\mathrm{d}t^2} = a^2 - \frac{c}{b}\,, \tag{1}$$

subject to the constraint

$$\frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\mathrm{d}\ln b}{\mathrm{d}t} + \frac{\mathrm{d}\ln a}{\mathrm{d}t}\frac{\mathrm{d}\ln c}{\mathrm{d}t} + \frac{\mathrm{d}\ln b}{\mathrm{d}t}\frac{\mathrm{d}\ln c}{\mathrm{d}t} = a^2 + \frac{b}{a} + \frac{c}{b}\,, \tag{2}$$

where $a = a(t)$, $b = b(t)$, and $c = c(t)$ are the so-called directional scale factors, while t is a monotonic function of proper time.

It has been found in [10] that the analytical solution to Eqs. (1)–(2), for $t > t_0$, reads

$$a(t) = \frac{3}{t - t_0}\,, \qquad b(t) = \frac{30}{(t-t_0)^3}\,, \qquad c(t) = \frac{120}{(t-t_0)^5}\,, \tag{3}$$

where $t - t_0 \neq 0$ and t_0 is an arbitrary real number.

The stability analyses carried out in [10] have shown that solution (3) is unstable against small perturbation

$$a(t) \;=\; 3(t-t_0)^{-1} + \epsilon\alpha(t) =: \tilde{a}(t) + \epsilon\alpha(t)\,, \tag{4a}$$
$$b(t) \;=\; 30(t-t_0)^{-3} + \epsilon\beta(t) =: \tilde{b}(t) + \epsilon\beta(t)\,, \tag{4b}$$
$$c(t) \;=\; 120(t-t_0)^{-5} + \epsilon\gamma(t) =: \tilde{c}(t) + \epsilon\gamma(t)\,. \tag{4c}$$

We have found [10] that the perturbation can be parameterized by five constants defining a submanifold of $\mathbb{R}^5$ with non-zero measure so that it is generic.

The relative perturbations $\alpha/a, \beta/b$, and γ/c grow proportionally as $\exp(\frac{1}{2}\theta)$, where $\theta = \ln(t - t_0)$. The multiplier 1/2 plays the role of a Lyapunov exponent, describing the rate of their divergence. Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) is chaotic.

The chaos results from the strong non-linearity of the dynamics and growing curvature of spacetime increasing effectively the non-linearity in the evolution towards the singularity.

3. Quantization of the BKL scenario

In what follows, we quantize the BKL scenario by using the integral quantization method called the affine coherent states quantization (see, *e.g.*, [13] and references therein).

We have already quantized Hamilton's dynamics of that scenario ignoring its chaotic phase: quantum singularity turns into quantum bounce and quantum evolution is unitary across quantum bounce [14, 15]. In the quantization of the chaotic phase of the BKL scenario, we do not quantize Hamilton's dynamics, but the solution to the BKL scenario, and we quantize both temporal and spatial variables to support general covariance of general relativity.

In the standard quantum mechanics time is not considered to be a quantum observable, but a parameter enumerating events. In this paper, we treat the time and position on the same footing at quantum level. This idea requires introducing the notion of an extended classical configuration space including time as an additional variable.

In what follows, we use the results of our paper [12].

3.1. Configuration space

Definition of the configuration space

$$T = \{\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) : \xi \in (\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+)\}\,, \quad (5)$$

where each pair (ξ_k, ξ_{k+1}) (where $k = 1, 3, 5$) defines a half-plane. It is known that the half-plane can be identified with the affine group $\mathrm{Aff}(\mathbb{R})$.

The scale factors are denoted as follows: $\xi_2 = a$, $\xi_4 = b$, $\xi_6 = c$. Because $a, b, c > 0$ and $\xi_1, \xi_3, \xi_5 \in \mathbb{R}$, the configuration space parameterizes the simple product of 3 affine groups $\mathrm{Aff}(\mathbb{R}) \times \mathrm{Aff}(\mathbb{R}) \times \mathrm{Aff}(\mathbb{R}) =: G$ to be used in quantization.

As the observational data are parameterized by a single time parameter, the variables $\{\xi_1,\, \xi_3,\, \xi_5\}$ should be mapped onto a single variable representing time.

3.2. Hilbert space

The direct product of three affine groups G has the unitary irreducible representation in the following Hilbert space: $\mathcal{H} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2} \otimes \mathcal{H}_{x_3} = L^2(\mathbb{R}^3_+, \mathrm{d}\nu(x_1,x_2,x_3))$, where $\mathrm{d}\nu(x_1,x_2,x_3) = \mathrm{d}\nu(x_1)\mathrm{d}\nu(x_2)\mathrm{d}\nu(x_3)$.

It enables defining in $\mathcal{H}$ the continuous family of affine coherent states $\langle x_1,x_2,x_3|\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6\rangle := \langle x_1|\xi_1,\xi_2\rangle\langle x_2|\xi_3,\xi_4\rangle\langle x_3|\xi_5,\xi_6\rangle$, as follows:

$$\mathcal{H} \ni \langle x_1,x_2,x_3|\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6\rangle := U(\xi)\Phi_0(x_1,x_2,x_3)\,, \tag{6}$$

where $U(\xi) := U(\xi_1,\xi_2)U(\xi_3,\xi_4)U(\xi_5,\xi_6)$ and $|\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6\rangle := |\xi_1,\xi_2\rangle\, |\xi_3,\xi_4\rangle|\xi_5,\xi_6\rangle$, and where

$$\mathcal{H} \ni \Phi_0(x_1,x_2,x_3) = \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\,. \tag{7}$$

3.3. Quantum observables

The resolution of the identity in $\mathcal{H}$ can be used for mapping a classical observable $f : T \to \mathbb{R}$ onto an operator $\hat{f} : \mathcal{H} \to \mathcal{H}$ as follows [13]:

$$\begin{aligned}\hat{f} &:= \frac{1}{A_\phi}\int_G \mathrm{d}\mu(\xi)|\xi\rangle f(\xi)\langle\xi| \\ &= \frac{1}{A_{\Phi_1}A_{\Phi_3}A_{\Phi_5}}\int_{\mathrm{Aff}(\mathbb{R})}\mathrm{d}\mu(\xi_1,\xi_2)\int_{\mathrm{Aff}(\mathbb{R})}\mathrm{d}\mu(\xi_3,\xi_4)\int_{\mathrm{Aff}(\mathbb{R})}\mathrm{d}\mu(\xi_5,\xi_6) \\ &\quad\times|\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6\rangle f(\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6)\langle\xi_1,\xi_2;\xi_3,\xi_4;\xi_5,\xi_6|\,.\end{aligned} \tag{8}$$

There exist two important characteristics of quantum observable: *(i)* expectation value — which corresponds to classical values of measured observable, and *(ii)* variance — which describes quantum smearing of observable.

3.4. Quantum dynamics

Quantum states, $\Psi_\tau(x_1,x_2,x_3) = \langle x_1,x_2,x_3|\Psi_\tau\rangle$, must be elements of the Hilbert space $\mathcal{H}$. Subscript τ labels quantum states, and it should be a one-to-one monotonic function of classical time t.

We require the states $|\Psi_\tau\rangle$ to satisfy the following conditions:

$$\begin{aligned}\langle\Psi_\tau|\hat{\xi}_k|\Psi_\tau\rangle &= t\,, \qquad k=1,3,5 &(9)\\ \langle\Psi_\tau|\hat{\xi}_2|\Psi_\tau\rangle &= a(t)\,, &(10)\\ \langle\Psi_\tau|\hat{\xi}_4|\Psi_\tau\rangle &= b(t)\,, &(11)\\ \langle\Psi_\tau|\hat{\xi}_6|\Psi_\tau\rangle &= c(t)\,. &(12)\end{aligned}$$

Equation (9) represents the single time constraint. Equations (9)–(12) define the quantum equations of motion. They relate the quantum dynamics to the classical one.

3.5. Evolving wave packets

In what follows, we use the Gaussian distribution wave packets

$$\Psi_n(x;\tau,\gamma) = N x^n \exp\left[i\tau x - \frac{\gamma^2 x^2}{2}\right]\,, \qquad N^2 = \frac{2\gamma^n}{(n-1)!}\,, \tag{13}$$

which are dense in $L^2(\mathbb{R}_+, \mathrm{d}\nu(x))$. Expectation values and variances of $\hat{\xi}_k$ and $\hat{\xi}_{k+1}$ are

$$\begin{aligned}\langle\Psi_n|\hat{\xi}_k|\Psi_n\rangle &= \tau\,, \qquad k=1,3,5\,, &(14)\\ \langle\Psi_n|\hat{\xi}_{k+1}|\Psi_n\rangle &= \frac{1}{A_\Phi}\frac{\Gamma\left(n-\frac{1}{2}\right)}{(n-1)!}\,\gamma\,, &(15)\\ \mathrm{var}\left(\hat{\xi}_k;\Psi_n\right) &= \frac{4n-3}{4(n-1)}\,\gamma^2\,, &(16)\\ \mathrm{var}\left(\hat{\xi}_{k+1};\Psi_n\right) &= \frac{1}{A_\Phi^2}\left(\frac{1}{n-1} - \frac{\Gamma\left(n-\frac{1}{2}\right)^2}{(n-1)!^2}\right)\gamma^2\,. &(17)\end{aligned}$$

In the space $L^2(\mathbb{R}_+^3, \mathrm{d}\nu(x_1,x_2,x_3))$, we take the corresponding wave packets

$$\begin{aligned}&\Psi_{n_1,n_3,n_5}(x_1,x_2,x_3;\,\tau_1,\tau_3,\tau_5,\gamma_1,\gamma_3,\gamma_5)\\ &= \Psi_{n_1}(x_1;\tau_1,\gamma_1)\Psi_{n_3}(x_2;\tau_3,\gamma_3)\Psi_{n_5}(x_3;\tau_5,\gamma_5)\,.\end{aligned} \tag{18}$$

To meet the properties of (9)–(12) for the wave packets Ψ_{n_1,n_3,n_5}, we choose the parameters τ_k and γ_k as follows:

$$\begin{aligned}\tau_1 &= \tau_3 = \tau_5 = t\,, &(19)\\ \gamma_k &= A_{\Phi_k}\frac{(n_k-1)!}{\Gamma\left(n_k-\frac{1}{2}\right)}\,f_k(t)\,, \qquad k=1,3,5\,, &(20)\end{aligned}$$

where

$$f_k(t) = \begin{cases} \tilde{a}(t) + \epsilon\alpha(t)\,, & k = 1\,, \\ \tilde{b}(t) + \epsilon\beta(t)\,, & k = 3\,, \\ \tilde{c}(t) + \epsilon\gamma(t)\,, & k = 5\,. \end{cases} \tag{21}$$

Variances in the Hilbert space $\mathcal{H}$ for the Gaussian wave packets read

$$\mathrm{var}\left(\hat{\xi}_k; \Psi_{n_1,n_3,n_5}\right) = \mathcal{A}_k f_k(t)^2\,, \tag{22}$$

$$\mathrm{var}\left(\hat{\xi}_{k+1}; \Psi_{n_1,n_3,n_5}\right) = \mathcal{B}_k f_k(t)^2\,, \tag{23}$$

where

$$\mathcal{A}_k = A^2_{\Phi_k} \frac{(4n_k - 3)(n_k - 1)!(n_k - 2)!}{4\Gamma\left(n_k - \frac{1}{2}\right)^2}\,, \tag{24}$$

$$\mathcal{B}_k = \frac{(n_k - 1)!(n_k - 2)!}{\Gamma\left(n_k - \frac{1}{2}\right)^2} - 1\,. \tag{25}$$

These results show that all positions of our system in time and space are smeared owing to non-zero variances. It is an important fact about possibility of avoiding singularities in this dynamics.

4. Stochasticity of quantum BKL scenario

Having calculated the variances of quantum observables corresponding to perturbed $\{a, b, c\}$ and unperturbed $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ solutions, we describe the quantum instabilities as follows:

$$\kappa_k := \frac{\mathrm{var}\left(\hat{\xi}_k; \Psi_{\mathrm{pert}}\right) - \mathrm{var}\left(\hat{\xi}_k; \Psi_{\mathrm{unpert}}\right)}{\mathrm{var}\left(\hat{\xi}_k; \Psi_{\mathrm{unpert}}\right)}\,, \qquad k = 2, 4, 6\,, \tag{26}$$

where $\hat{\xi}_2 = \hat{a}$, $\hat{\xi}_4 = \hat{b}$, $\hat{\xi}_6 = \hat{c}$, and where Ψ_{pert} and Ψ_{unpert} denote perturbed and unperturbed wave packets, respectively.

Making use of

$$\begin{aligned} f_2(t)^2 &= (\tilde{a}(t) + \epsilon\alpha(t))^2 = \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t) + \epsilon^2\alpha(t)^2 \simeq \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t)\,, \\ f_4(t)^2 &= (\tilde{b}(t) + \epsilon\beta(t))^2 = \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t) + \epsilon^2\beta(t)^2 \simeq \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t)\,, \\ f_6(t)^2 &= (\tilde{c}(t) + \epsilon\gamma(t))^2 = \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t) + \epsilon^2\gamma(t)^2 \simeq \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t)\,. \end{aligned}$$

we obtain explicit form of (26), which in the 1^{st} order in ϵ, reads

$$\kappa_a(t) := \kappa_2(t) = \frac{2\epsilon\tilde{a}(t)\alpha(t)}{\tilde{a}(t)^2} = 2\epsilon\frac{\alpha(t)}{\tilde{a}(t)}\,, \tag{27}$$

$$\kappa_b(t) := \kappa_4(t) = \frac{2\epsilon\tilde{b}(t)\beta(t)}{\tilde{b}(t)^2} = 2\epsilon\frac{\beta(t)}{\tilde{b}(t)}\,, \tag{28}$$

$$\kappa_c(t) := \kappa_6(t) = \frac{2\epsilon\tilde{c}(t)\gamma(t)}{\tilde{c}(t)^2} = 2\epsilon\frac{\gamma(t)}{\tilde{c}(t)}\,. \tag{29}$$

Figure 1 presents the parametric curve visualizing the relative quantum perturbations. Higher-order approximations in ϵ would not change much that curve.

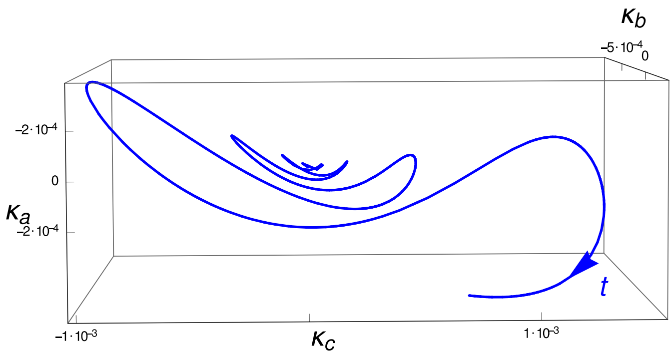

Fig. 1. The t dependence of quantum perturbation defined by (27)–(29) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\phi_1 = \phi_2 = 0$, $\epsilon = 0.01$. The plot presents the parametric curve $\{\kappa_a(t), \kappa_b(t), \kappa_c(t)\}$, where $t \in (0.01, 35)$.

5. Conclusions

Since our quantum and classical perturbations have quite similar time evolutions (see, Fig. 2 of [10]), we conclude that quantization does not destroy classical chaos. In fact, quantum chaos corresponds to classical chaos in the lowest-order approximation. Non-linearity of classical dynamics creates deterministic chaos. Non-vanishing variances of observables of the corresponding quantum dynamics lead to stochastic chaos.

As calculated variances are always non-zero, the probability of obtaining divergencies of quantum observables corresponding to classical gravitational singularity is equal to zero, which is consistent with the results of [14, 15].

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