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INTEGRAL QUANTIZATION AND QUANTUM TIME*

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Integral quantization and time treated on the same footing as other quantum observables are considered. They allow to construct quantum gravity models in a more natural way because an idea of time as a quantum observable is consistent with General Relativity, contrary to time treated as a parameter. The projection evolution formalism is shortly presented. A semiquantal approximation is defined.

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1. Introduction

Quantization is a kind of art that allows deriving quantum observables from their classical counterparts. There are many quantization methods which are to some extent equivalent, *i.e.*, usually, they produce the same results only for a small family of quantum systems.

Quantization can be performed either on a phase space which doubles required physical degrees of freedom introducing pairs of canonically conjugated variables or on the configuration space whose dimension is equal to the number of physical degrees of freedom.

In the standard approach to mechanics, time is not considered as a physical degree of freedom of either the classical or quantum system.

However, classical and quantum relativity strongly suggest including time (and eventually its canonically conjugated observable the temporal momentum) as a new variable that has to be treated on the same footing as other variables describing the physical system [1].

In the standard approach, the most popular is the canonical quantization which maps a phase space of classical mechanics (CM) to a set of quantum observables in quantum mechanics (QM). According to this approach, the map denoted $\Theta : \mathrm{CM} \to \mathrm{QM}$ should fulfill the following condition:

$$\Theta(\{f,g\}) = \frac{1}{i\hbar}[\Theta(f),\Theta(g)]\,. \tag{1}$$

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This quantization is a subject of the Groenewold theorem [2] which says that there is no reasonable quantization map satisfying the above identity exactly for all classical functions f, g. Another problem is the lack of a unique ordering method of operators during the quantization process.

A more flexible method of quantization is a generalization of the canonical quantization by introducing the so-called star product

$$f \star g \;=\; f \cdot g + (i\hbar) C_1(f,g) + \sum_{n=2}^{\infty} (i\hbar)^n C_n(f,g)\,, \tag{2}$$

$$C_1(f,g) \;=\; \frac{1}{2}\{f,g\}\,, \tag{3}$$

$$\Theta(f) \cdot \Theta(g) \;=\; \Theta(f \star g)\,. \tag{4}$$

Different choices of the $\star$-products cover a large class of quantization methods. In this case, the mapping $\Theta : \mathrm{CM} \to \mathrm{QM}$ fulfills condition (4). The Groenewold objection and the ordering problem are still valid. For a review, see [3].

On the market, one can find a huge amount of very different quantization methods such as: Green function quantization, path integrals, Wigner function quantization, algebraic quantizations, perturbatively quantized gravity, geometrodynamical canonical quantization, the Wheeler–Dewitt equation, the Euclidean path integral the approach of Hawking, Penrose twistor theory, string theory, asymptotically safe gravity, causal dynamical triangulation, emergent gravity, loop quantum gravity, *etc.* This is only a small part of all quantization approaches.

In this paper, we describe a simplified version of a special quantization technique which can be named a deformation of a quantum measure and applied to the quantization of time or more exactly spacetime. The more sophisticated and more universal method is based on group algebras and GNS construction [4].

2. Integral quantization

Let us consider an overcomplete set of vectors $|g\rangle = U(g)|\Phi_0\rangle$, where $U(g)$ is a unitary and irreducible representation of a locally compact group G in a Hilbert space $\mathcal{K}$. The so-called fiducial vector $|\Phi_0\rangle$ is constructed to get the orbit $\mathcal{O}_{\mathrm{G}} = \{U(g)|\Phi_0\rangle : g \in \mathrm{G}\}$ as a dense set in $\mathcal{K}$. In this case, the vectors $|g\rangle$ represent the non-orthogonal resolution of unity

$$\int_{\mathrm{G}} \mathrm{d}\mu(g) |g\rangle\,\langle g| = \hat{\mathbf{1}}\,. \tag{5}$$

This means that the operators

$$\hat{M}(\Omega) := \int_{\Omega} \mathrm{d}\mu(g)|g\rangle\langle g| \tag{6}$$

form a positive operator valued measure (POVM). The special case of operators (6) is

$$\hat{M}(g) = |g\rangle\langle g|\,, \tag{7}$$

where $g \in \mathrm{G}$, represents the projection onto the state $|g\rangle$.

A link between classical functions and quantum states is given by the group G. The group G is considered as an extended configuration space of both the classical and the corresponding quantum systems. It contains time as a variable.

In the following, we use the notion of configuration space but the same considerations can be done for the extended phase space (such phase space contains time and the temporal component of linear momentum as the additional degrees of freedom).

Because of (6), the elementary vectors $|g\rangle$ (we call them quantum points), where $g \in \mathrm{G}$, can be interpreted as the quantum states corresponding to the appropriate configuration points. According to our assumption, the space-time parametrization is a part of the full parametrization of the group G. It means, the scalar product $\langle g_2|g_1\rangle$ represents the transition probability amplitude between quantum points $|g_1\rangle \to |g_2\rangle$.

The set of vectors $\mathcal{O}_{\mathrm{G}}$ represents the degenerate vacuum state of the configuration space.

This interpretation requires that the quantum points $|g\rangle$ should be a subset of preferred states of the quantum system under consideration. They represent the vacuum as a physical object. All other states should be "constructed" from these quantum points.

One sees that, because of (5), every state $|\Psi\rangle$ can be expressed as a generalized linear combination of the quantum points $|g\rangle$

$$|\Psi\rangle = \int_{\mathrm{G}} \mathrm{d}\mu(g)|g\rangle\langle g|\Psi\rangle\,. \tag{8}$$

Following the above interpretation, the function

$$\mathrm{Prob}\,(\Omega) = \mathrm{Tr}\left[\hat{M}(\Omega)\rho\right]\,, \tag{9}$$

where ρ is a density operator representing the state of the system, gives the probability that our quantum system is in the region Ω of the configuration space. In the special case of the POV operators (7), one gets the probability of being in the point g as $\mathrm{Prob}\,(g) = \mathrm{Tr}[\hat{M}(g)\rho]$. Since $\int_{\mathrm{G}} \mathrm{d}\mu(g)\mathrm{Prob}\,(g) = 1$, the function $\mathrm{Prob}\,(g)$ is, in fact, a density probability.

Calculation of the expectation value of any classical function $f(g)$ representing classical observable with the quantum probability $\mathrm{Prob}\,(g)$ gives

$$\langle f\rangle = \int\limits_{\mathrm{G}} \mathrm{d}\mu(g) f(g) \mathrm{Prob}\,(g) = \mathrm{Tr}\left[\left(\int\limits_{\mathrm{G}} \mathrm{d}\mu(g)|g\rangle f(g)\langle g|\right)\rho\right], \tag{10}$$

for all quantum states represented by the density operators ρ.

This suggests that the quantized version of the classical observable $f(g)$ should be given by the operator

$$\hat{f} := \int\limits_{\mathrm{G}} d\mu(g)|g\rangle f(g)\langle g|\,. \tag{11}$$

This quantization is dependent on the map between the classical configuration space and its representation in quantum description [5, 6].

3. Time as a quantum observable

Quantum gravity and also quantization of gravity require time to be considered on the same footing as other space observables, *i.e.*, the quantum time should be a part of the spacetime position quantum observable. A standard approach to quantum mechanics is not appropriate in this case because time is treated there as a parameter, not a variable. A possible solution is the formalism of the projection evolution described in [1].

The main assumption of this formalism is the *changes principle*:

> The evolution of a quantum system is a random process of choosing the next step of the physical system caused by spontaneous changes in the Universe.

To perform this stochastic process (evolution) of a given quantum system, one needs to build the projection evolution operators. The projection evolution operators $\mathbb{F}$ at the evolution step τ_n are defined as a family of allowed transformations between quantum states, labeled by sets of the quantum numbers ν, from the evolution step τ_{n-1} to the evolution step τ_n

$$\rho(\tau_n;\nu') = \mathbb{F}(\tau_n;\nu',\rho(\tau_{n-1};\nu))\,, \tag{12}$$

where the operator $\rho(\tau_k;\mu)$ denotes a quantum state at the evolution step τ_k labeled by a set of the quantum numbers μ.

The parameter τ_n is only an ordering parameter enumerating subsequent changes in quantum states. It is a global and absolute and not measurable parameter, *i.e.*, it is not TIME.

The generalized Lüders projection postulate [7] is proposed as the principle for the evolution

$$\rho(\tau_n;\nu_n) = \frac{\digamma(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))}{\mathrm{Tr}\left(\digamma(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))\right)}\,. \tag{13}$$

The projection evolution is a stochastic process associated with a given probability distribution. The probability distribution for choosing a next state is given by a quantum mechanical transition probability from the previous to the next state, allowed by the projection postulate (13).

The procedure of choosing the next state of the considered physical system is called the chooser.

This formalism allows to consider time as a quantum observable because the evolution is parameterized by a formal ordering parameter τ which enumerates subsequent evolution steps. All required time characteristics of a physical system at a given evolution step can be calculated in the same way as remaining quantum observables — the scalar product in such state space contains integration over time. The most important quantum characteristics are expectation values of quantum observables.

According to expressions (10) and (11), the classical time is transformed into the operator $\hat{t}$. The expectation value of time in a physical system being in the state $\rho(\tau_n;\nu_n)$ obtained at the evolution step τ_n is

$$\bar{t}_n = \mathrm{Tr}\left[\left(\int\limits_{\mathrm{G}} \mathrm{d}\mu(g)|g\rangle\, t\, \langle g|\right)\rho(\tau_n;\nu_n)\right]\,. \tag{14}$$

In this way, one can obtain subsequent instants $\bar{t}_n$ characterizing evolving system. This allows to compare expectation values of every observable at a given $\bar{t}_n$ and compare it with the appropriate experimental values.

4. Semiquantal approach

In the case of a quantum system, the simplest approach allowing for a simplified type of such calculation is using the classical solutions of its equations of motion. This semiquantal approach in the extended configuration space can be described by the following steps:

- Identifying the following set of classical spacetime observables characteristic for the system under consideration:
  - the spacetime positions x^μ;
  - the N_R classical solutions of the equations of motions $\xi_l = \phi_l(\{x^\mu\})$ as functions of the spacetime variables, $l = 1, 2, \ldots, N_R$.

— Quantizing the positions $x^\mu \to \hat{x}^\mu$ and the functions $\xi_l = \phi_l(\{x^\mu\}) \to \hat{\xi}_l$.

— Choosing a family of trial states $|\Psi_\eta\rangle$ parameterized by a set of parameters η.

— Solving the following system of equations:

$$\langle\Psi_\eta|\hat{x}^\mu|\Psi_\eta\rangle = x^\mu\,, \tag{15}$$

$$\langle\Psi_\eta|\hat{\xi}_l|\Psi_\eta\rangle = \phi_l(\{x^\mu\}) \tag{16}$$

to determine the required approximate states of the system, *i.e.*, to find a set of the parameters η.

By solving these equations, one gets the quantum states $|\Psi_\eta\rangle$ reproducing classical solutions of the system, in average.

Examples of such solutions are shown in [6, 8].

One needs to notice that Eqs. (15) and (16) can have either many sets of solutions or no solutions at all. In the first case, one can construct many evolution paths, in the second case, one needs to extend the set of trial functions.

By choosing an evolution path $|\Psi_\eta\rangle$, one can calculate all required quantum characteristics of the system during its evolution.

The classical solutions of the required equations of motions are reproduced exactly as expectation values of the appropriate operators, however, usually they are smeared quantities. A measure of this smearing is given by their quantum variances. The nonzero variances prevent the corresponding quantum observables to be singular at the points where the classical observables have singularities.

Generally, one can see that the nonzero variances of quantum observables can change behavior of the quantum system drastically.

5. Conclusions

The integral quantization is a nonlocal-type quantization which allows to reproduce average values of classical observables calculated with a nearly arbitrary probability distribution (9). This method is free from the ordering problem. It allows to quantize nearly any kind of classical function, the limitation is done by a convergence of integrals involved in this procedure.

The integral quantization can be applied to function spaces created on any locally compact group. To have physical interpretation, the group manifold is mapped either to configuration or phase space of the required physical system. This quantization can be extended to much more general manifolds having the well-defined measure.

In this paper, we are using the notion of extended configuration space assuming time to be the additional degree of freedom of every physical system.

This assumption requires changes in the description of quantum evolution. The projection evolution approach is proposed (PEv). It allows to treat time on the same footing as remaining observables. This means that all temporal quantum characteristics of a quantum system have to be calculated.

Mathematical construction of the quantum evolution requires introducing an ordering parameter τ which enumerates subsequent evolution steps of the system under consideration. This parameter is only an ordering parameter which does not have any additional structure and is not a measurable quantity. For every step of the evolution, the expectation value of the time operator can be calculated. This fixes the classical time and the quantum system state corresponding to this time. Having the state of our system at the classical time $\bar{t}$ calculation of expectation values, variances, and potentially higher-order statistical moments of required quantum observables allow to characterize our quantum system completely.

The semiquantal approach described in the last section is an approximation which allows to use a very simplified version of the PEv idea. It reproduces classical solutions as expectation values of the corresponding observables. In fact, this describes the worst quantum scenario in which most of the classical singularities are reproduced. However, the calculation of variances of these singular observables allows to check if the quantum system is singular or the singularity is smeared out. The nonzero variances diminish the probability of falling into a singularity substantially — formally, the probability of falling into the pointlike singularity is equal to zero.

It seems that this mechanism is a promising way of explaining how quantum mechanics allows to avoid singularities in the quantum dynamics of gravitational systems.

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