ASCRIBING QUANTUM SYSTEM TO SCHWARZSCHILD SPACETIME WITH NAKED SINGULARITY — ACS QUANTIZATION METHOD*

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The quantization of the Schwarzschild black hole by using the affine coherent state (ACS) quantization method is presented. I introduce quantization of both temporal and spatial coordinates. I propose the method of quantum analysis of the gravitational singularity. In the presented model, the quantum effects smear the gravitational singularity indicated by the Kretschmann invariant avoiding its localization in the configuration space.

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1. Introduction

The affine coherent state (ACS) quantization method is an effective quantization method convenient for the construction of a quantum spherical symmetric gravitational model [1–3]. In such a model, the angles are not the dynamical variables, and the configuration space sufficient to describe interesting gravitational characteristics can be reduced to the form

$$T = \{(t, r) \mid (t, r) \in \mathbb{R} \times \mathbb{R}_+\}, \tag{1}$$

where t is time and r is radial variable.

The quantization procedure is based on correspondence between the points from the configuration space and quantum projection operators. The projection operators can be interpreted as operators which are mapping from the configuration space to the quantum space. In this paper, I present an application of the ACS quantization method in the Schwarzchild space-time.

A very interesting feature of this quantization method is a possibility of constructing of temporal variables. The equivalent treatment of both the temporal and spatial variables is the basic fact underlying the whole

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relativistic physics. In the presented approach to the quantization of gravitational systems, I apply this property to the quantum picture. In the ACS quantization method, the construction of the position and time operators is a very natural task.

Another interesting problem, which is common to gravity quantization projects, is a quantum analysis of gravitational singularities. One expects that a theory of quantum gravity is a fundamental theory which has to be applied in a close neighborhood of classical singular points. Therefore, the quantum effects are expected to be decisive in identifying the existence of a gravitational singularity.

The presented paper is based on [4].

2. ACS quantization method

The ACS quantization method is based on the affine groups. The elements of the affine group are parametrized by two parameters where the first one is a real number and the second one is a real positive number $(p,q) \in \mathbb{R} \times \mathbb{R}_+$. The multiplication law is taken as follows [3]:

$$g(p_1,q_1) \cdot g(p_2,q_2) := g(p_1 + q_1 p_2, q_1 q_2) \in \mathrm{Aff}(\mathbb{R})\,. \tag{2}$$

The left-hand side invariant measure is defined as

$$\mathrm{d}\mu(p,q) = \frac{1}{2\pi}\mathrm{d}p\frac{\mathrm{d}q}{q^2}\,. \tag{3}$$

According to the requirements of the quantization procedure, every point of the configuration space must be uniquely identified with the corresponding group element

$$(t,r) \leftrightarrow g(\chi(t,r)) = g(p,q)\,, \tag{4}$$

where $\chi(t,r)$ is one-to-one transformation.

As a quantum carrier space, we take the space of square integrable functions on the half-line $\mathcal{H}_x = L^2(\mathbb{R}_+, \mathrm{d}\nu(x))$, where the measure is given by $\mathrm{d}\nu(x) = \frac{\mathrm{d}x}{x}$. To associate the carrier space and affine group, one needs to take irreducible unitary representation of the affine group

$$U(p,q)\Psi(x) = \mathrm{e}^{ipx}\Psi(qx)\,. \tag{5}$$

The next step in the quantization procedure is to choose the fiducial vector $\Phi_0(x) \in \mathcal{H}_x$. By the action of Eq. (5) on the fiducial vector $\Phi_0(x) \in \mathcal{H}_x$, one gets the so-called coherent states

$$\langle x|g(p,q)\rangle = U(p,q)\Phi_0(x) = \mathrm{e}^{ipx}\Phi_0(qx)\,. \tag{6}$$

The fiducial vector has to be a normalized vector and the following integral must be satisfied [3]:

$$\langle \Phi_0 | \Phi_0 \rangle = \int_0^\infty \mathrm{d}\nu(x) |\Phi_0(x)|^2 = 1\,, \tag{7}$$

$$A_{\Phi_0} := \int_0^\infty \frac{\mathrm{d}x}{x^2} |\Phi_0(x)|^2 < \infty\,. \tag{8}$$

The fiducial vector must be selected to provide a resolution of unity

$$\frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(p,q)\ |g(p,q)\rangle\langle g(p,q)| = \hat{\mathbf{1}}\,. \tag{9}$$

The quantization procedure is based on scaling of every projection operator $|g\rangle\langle g|$ by the value of the classical observable $f(g)$ at the point g. Next, one needs to integrate these expressions over the full group manifold. Therefore, mapping of the real observable $f : T \to \mathbb{R}$ into a symmetric operator is performed in the following way [1, 3]:

$$\hat{f} := \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(p,q) |g(p,q)\rangle f(p,q) \langle g(p,q)|\,. \tag{10}$$

3. ACS quantization of elementary observables

Let the transformation between the configuration space and group elements have the following form:

$$\left\{ \begin{array}{l} \chi_1(t,r) = p \\ \chi_2(t,r) = q \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \chi_1^{-1}(p,q) = t \\ \chi_2^{-1}(p,q) = r \end{array} \right. .$$

By using the above correspondence and ACS quantization procedure, one can construct operators of all elementary observables

$$\hat{t} = \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(p,q) |g(p,q)\rangle \left(\chi_1^{-1}(p,q)\right) \langle g(p,q)|\,, \tag{11}$$

$$\hat{r} = \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(p,q) |g(p,q)\rangle \left(\chi_2^{-1}(p,q)\right) \langle g(p,q)|\,. \tag{12}$$

In the above constructions, the place of classical observables is taken by functions that correspond to observables t and r.

The freedom in choosing transformation χ causes difficulties in the interpretation of coherent states [4]. To fix this interpretation, one assumes that expectation values of the time operator $\hat{t}$ within the coherent states $|g(t,r)\rangle$ reproduce the classical time

$$\langle g(t,r)|\hat{t}|g(t,r)\rangle = t\,. \tag{13}$$

Similarly, the expectation value of the operator $\hat{r}$ in the state $|g(t,r)\rangle$ should be equal to the classical position in the space

$$\langle g(t,r)|\hat{r}|g(t,r)\rangle = r\,. \tag{14}$$

4. Analysis of gravitational singularity

An important quantum characteristic of observables and quantum states is the variance of this observable within such a state

$$\mathrm{var}\left(\hat{A};\psi\right) = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2\,. \tag{15}$$

The variance measures a spread of $\hat{A}$ around its expectation value.

Now let us turn to the problem of quantum analysis of gravitational singularity. In the first step, one has to assume a set of classical curvature invariants $\mathcal{A}_n(t,r)$ which characterize the gravitational model. After the ACS quantization of these observables, one gets a set of quantum operators $\hat{\mathcal{A}}_n$.

The quantum gravitational singularity in the state $|\psi\rangle$ is reached, if the following two conditions are satisfied:

1. The expectation values of the curvature invariants operators go to infinity in this state
$$\langle\psi|\hat{\mathcal{A}}_n|\psi\rangle \to \infty\,. \tag{16}$$
2. The variances of the curvature invariants operators go to 0
$$\mathrm{var}\left(\hat{\mathcal{A}}_n;\psi\right) \to 0\,. \tag{17}$$

The first condition provides a classical interpretation of these quantum observables in the state $|\psi\rangle$. They coincide with the standard classical description. The second condition provides that the states are "localized" exactly at the singular point and the probability of finding our system outside the singularity is equal to 0.

The Schwarzschild black hole is one of the simplest spherical symmetric vacuum solutions of Einstein's equations. The form of the Schwarzchild metric in the so-called Schwarzschild coordinates $(t, r, \theta, \phi) \in \mathbb{R} \times (0, \infty) \times S^2$ has the form [5, 6]

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right), \qquad (18)$$

In our considerations we take $M < 0$, so that it describes the naked singularity.

The curvature invariant which exhibits the gravitational singularity as $r \to 0$ is the Kretschmann scalar

$$\mathcal{K} := R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}\,.$$

Using the affine coherent state quantization method, the Kretschmann observable reads

$$\hat{\mathcal{K}} = 48M^2 \langle \check{q} \rangle_0^6 \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} \mathrm{d}\mu(p,q) |g(p,q)\rangle \frac{1}{q^6} \langle g(p,q)|\,,$$

where $\langle \check{q} \rangle_0 = \langle g(0,1)|\check{q}|g(0,1)\rangle$.

Since the coherent states are interpreted as the representatives of points in configuration space, it is interesting to analyze the expectation value and variance of the Kretschmann operator in these states

$$\langle g(t,r)|\hat{\mathcal{K}}|g(t,r)\rangle = 48M^2 \frac{\left\langle \left(q^{-6}\right)\check{}\,\right\rangle_0}{\langle \check{q} \rangle_0^{-6}} \frac{1}{r^6}\,, \qquad (19)$$

$$\mathrm{var}\left(\hat{\mathcal{K}}; g(t,r)\right) = \left(48M^2\right)^2 \left(\left\langle \left(\left(q^{-6}\right)\check{}\,\right)^2 \right\rangle_0 - \left\langle \left(q^{-6}\right)\check{}\,\right\rangle_0^2\right) \langle \check{q} \rangle_0^{12} \frac{1}{r^{12}}\,. \qquad (20)$$

The first condition of quantum gravitational singularity analysis (16) is true for the $\hat{\mathcal{K}}$ operator if r goes to 0. But in such a case, the variance of the Krechmann operator also goes to infinity (17). The above situation indicates that the coherent state with r close to 0 are a good candidate for being a state for which gravitational singularity is achieved. But the effect of quantum smearing prevents the system to be well-localized in gravitational singularity.

However, the above analysis does not give us a response to the general question if there exists a quantum state which reaches gravitation singularity. To answer this question, one can analyze the following set of functions:

$$\Psi_n(x) = N x^n \exp\left[i\tau_0 x - \frac{\gamma^2 x^2}{2}\right], \qquad N = 2\gamma^n/(n-1)! \qquad (21)$$

$\Psi_n(x)$ form a set of functions which is dense in the Hilbert space $L^2(\mathbb{R}_+, \mathrm{d}\nu(x))$.

The expectation values of the operators $\hat{t}$, $\hat{r}$, $\hat{\mathcal{K}}$ and variance of $\hat{\mathcal{K}}$ in the states Ψ_n are as follows (Fig. 1):

$$\langle \Psi_n|\hat{t}|\Psi_n\rangle = \tau_0 \,, \tag{22}$$

$$\langle \Psi_n|\hat{r}|\Psi_n\rangle = \frac{1}{A_\Phi}\frac{\Gamma\left(n-\frac{1}{2}\right)}{(n-1)!}\,\gamma \,, \tag{23}$$

$$\langle \Psi_n|\hat{\mathcal{K}}|\Psi_n\rangle = \mathcal{A}\frac{(n+2)!}{(n-1)!}\,\frac{1}{\gamma^6} \,, \tag{24}$$

$$\mathrm{var}\left(\hat{\mathcal{K}};\Psi_n\right) = \mathcal{A}^2\left(\frac{(n+5)!}{(n-1)!}-\frac{(n+2)!^2}{(n-1)!^2}\right)\frac{1}{\gamma^{12}} \,, \tag{25}$$

where $\mathcal{A}=\frac{48M^2\langle\check{q}\rangle_0^6}{A_{\Phi_0}}\int_{\mathbb{R}_+}\frac{\mathrm{d}q}{q^8}\,|\Phi_0(q)|^2$.

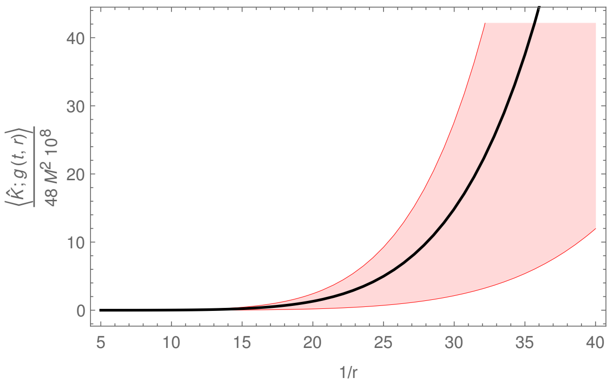

Fig. 1. The $1/r$ dependence of the expectation value of the Kretschmann operator $\langle g(t,r)|\hat{\mathcal{K}}|g(t,r)\rangle$. The red area defines the points for which the distance from the expected value is smaller than $\sqrt{\mathrm{var}(\hat{\mathcal{K}};g(t,r))}$ (the distance is counted along the fixed $1/r$ line). The fiducial vector is taken as $\Phi_0(x)=\frac{1}{\sqrt{(2n-1)!}}x^n\,\mathrm{e}^{-\frac{x}{2}}$ with $n=25$.

The above results show that the relation between the expectation value of observable r, t, and $\mathcal{K}$, and the variance of $\mathcal{K}$ are the same as for the coherent states and, therefore, we can get the same conclusions. Moreover, it can be shown that the same relationship occurs for nondiagonal matrix elements $n\neq m$

$$\langle \Psi_n|\hat{r}|\Psi_m\rangle \sim \gamma \,, \qquad \langle \Psi_n|\hat{\mathcal{K}}|\Psi_m\rangle \sim \frac{1}{\gamma^6} \,, \qquad \langle \Psi_n|\hat{\mathcal{K}}^2|\Psi_m\rangle \sim \frac{1}{\gamma^{12}} \,, \tag{26}$$

where element $\langle \Psi_n|\hat{\mathcal{K}}^2|\Psi_m\rangle$ is necessary for the calculation of variance (15). It proves that the same relation occurs for any linear combination of functions Ψ_n and, therefore, for any quantum states from the carrier space.

One can conclude that in the ACS quantum model of the Schwarzschild space-time, there is no such quantum state which achieves gravitational singularity.

5. Conclusions

At the first point in the summary, it should be highlighted the simplicity of the ACS quantization method which allows for qualitative analysis of gravitation models in sensitive areas.

The ACS quantization method naturally leads to the operator of quantum time observable which is significant in the program of quantization of gravity and which is worth being applied to more realistic models.

Making use of the ACS quantization of the Schwarzschild spacetime, we have found that the expectation value of the Kretschmann operator $\hat{\mathcal{K}}$ is singular and behaves like $1/r^6$ as in the classical case. However, its variance behaves like $1/r^{12}$. One can say that quantization smears the singularity, avoiding its localization in the region of the configuration space including the singularity.

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REFERENCES

[1] A. Góźdź, W. Piechocki, G. Plewa, *Eur. Phys. J. C* **79**, 45 (2019).

[2] A. Góźdź, W. Piechocki, *Eur. Phys. J. C* **80**, 142 (2020).

[3] A. Góźdź, W. Piechocki, T. Schmitz, *Eur. Phys. J. Plus* **136**, 18 (2021).

[4] A. Góźdź, A. Pędrak, W. Piechocki, *Class. Quantum Grav.* **39**, 145005 (2022).

[5] K. Schwarzschild, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* **1916**, 189 (1916), arXiv:physics/9905030, translation and foreword by S. Antoci and A. Loinger.

[6] J. Droste, «The field of a single centre in Einstein's theory of gravitation and the motion of a particle in that field», *K. Ned. Akad. Wet. Proc.* **19**, 197 (1917).