QUANTUM DYNAMICS OF RELATIVISTIC SYSTEMS*

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Received 7 February 2023, accepted 10 May 2023, published online 13 June 2023

We discuss the time problem in quantum gravity. We illustrate the problem with a model of gravitational waves in a quantum Friedmann universe. We propose a possible prescription for dealing with the unitarily inequivalent quantum dynamical descriptions based on different internal clocks. Our prescription permits unambiguous clock-independent physical predictions.

DOI:10.5506/APhysPolBSupp.16.6-A24

1. Introduction

Attempts at quantization of General Relativity notoriously suffer from the so-called time problem [1–3]. Let us first explain the nature of the problem. The diffeomorphism invariance in the canonical formulation [4] of the Einstein gravity leads to the Hamiltonian that is a constraint, $\boldsymbol{H}_G \approx 0$. Therefore, the quantization of the Hamiltonian and the imposition of the Hamiltonian constraint on the states of the gravitational system, $\widehat{\boldsymbol{H}}_{\mathrm{G}}|\Psi\rangle = 0$, leads to a "timeless dynamics". Indeed, there is no external parameter t called time, like in the usual Schrödinger equation $\widehat{\boldsymbol{H}}|\Psi\rangle = i\hbar\partial_t|\Psi\rangle$. One might, however, promote one of the internal degrees of freedom to the role of "internal clock" with respect to which the quantum evolution of the remaining variables could be followed. For instance, the quantum constraint equation could be given in the form of the Schrödinger equation $\widehat{\boldsymbol{H}}_{\mathrm{G}}|\Psi(q_1,\dots,q_n)\rangle \equiv -i\hbar\partial_{q_1}|\Psi(q_1,\dots,q_n)\rangle + \widehat{h}_{\mathrm{G}}|\Psi(q_1,\dots,q_n)\rangle$ together with the identification $t := q_1$. This looks like a reasonable way out of the conundrum. Unfortunately, one has to deal with potentially infinitely many internal clocks and the respective quantum dynamics they describe. These dynamics are unitarily inequivalent [5–9]. The simplest way to see it is by realizing that the choice of internal clock $t := q_1$ is accompanied by a redefinition of the scalar product $\langle\Psi|\Phi\rangle \to \langle\Psi|\delta(q_1 - t)|\Phi\rangle$. Since different

* Presented at the 8th Conference of the Polish Society on Relativity, Warsaw, Poland, 19–23 September, 2022.

choices of internal clock result in different scalar products, the properties of the states as well as of the operators will significantly differ between these different Hilbert spaces.

In the literature, one may come across various positions in regard to this fact. Some researchers believe that we cannot compare different internal clocks at the quantum level [10]. Others believe that the differences between different internal clocks could be perhaps described, nevertheless, they anyway must not be too large or quantum models of gravity would not make any sense [11]. Still, others find the discrepancies to be in fact very large and this is exactly where the time problem remains insufferably unsolved.

In this contribution, we show that internal clocks can be compared at the quantum level, or, more precisely, at the semi-classical level where the complications of the operator analysis in the full quantum description are avoided. Furthermore, we clearly show that the discrepancies between different clocks can be indeed huge contrary to the expectations of some researchers. Finally, we show how unambiguous (though restricted) predictions can be made with the use of any of these clocks. For the sake of discussion, we employ an important and integrable model of perturbations in a quantum universe, *i.e.*, quantum gravity waves propagating across a quantum Friedmann universe.

2. Clock transformations

The basic tool for making comparisons between internal clocks at the quantum level are the so-called clock transformations [7, 8, 12, 13]. The basic assumption we make is that the time problem is independent of whether first the constraint is quantized and then imposed on quantum states (Dirac's quantization) or first, the constraint is solved classically and then the obtained reduced phase space is quantized (the reduced phase-space quantization). On the one hand, in both approaches the internal clock must be eventually chosen and in some cases both methods can even yield identical results. On the other hand, the reduced phase-space quantization is usually more tractable. Therefore, in our study we use the latter.

The clock transformations are used to generate reduced phase spaces for a given gravitational system, which is equipped with a Hamiltonian generating dynamics with respect to different internal clocks. It should be noted that the form of the Hamiltonian can be transformed upon time-dependent canonical transformations in the reduced phase space. However, the particular form of the Hamiltonian is not relevant in our method and thus we shall always assume its most simple form. Now, suppose we have reduced a gravitational system with respect to an internal clock, denoted by T, and obtained the physical phase space equipped with a Hamiltonian H of a simple form and with a contact form $\omega_{\mathcal{C}}$

$$\omega_{\mathcal{C}} = \mathrm{d}q\,\mathrm{d}p - \mathrm{d}T\,\mathrm{d}H\,, \tag{1}$$

where (q, p) is the reduced phase space and $H(q, p)$ is the Hamiltonian generating the dynamics in the clock T. As already mentioned, what we are looking for is not any canonical transformation that changes the form of the Hamiltonian H. Rather, it is a clock transformation that changes $T \to T' = T + \varDelta(q, p, T)$, where $\varDelta$ is called the delay function. It is a wonderful property of integrable systems that we can find new phase-space variables (q', p') such that the contact form reads

$$\omega_{\mathcal{C}} = \mathrm{d}q'\,\mathrm{d}p' - \mathrm{d}T'\,\mathrm{d}H'\,, \tag{2}$$

where $H'(q', p') = H(q', p')$, *i.e., the form of the Hamiltonian remains the same.* It is a convenient property because once we solve the dynamics for q and p in the clock T, we get for free the dynamics of q' and p' in the clock T'. There is an even more important property of this transformation which we describe below. But first, let us find this transformation explicitly.

We have assumed above that the Hamiltonian H is clock-independent. In other words, it is a constant of motion whose form we require to be preserved upon the clock transformation. We find it natural to demand the same from all constants of motion, that is, their form be preserved under the clock transformation. For a $2n$-dimensional system, we assume $2n$ constants of motion denoted by H_J with $H_1 = H$. The clock transformations $(q, p, T) \mapsto (q', p', T')$ are then given as follows:

$$T' = T + \varDelta(q, p, T)\,, \qquad H_J(T, q, p) = H_J(T', q', p')\,, \qquad J = 1, \ldots, 2n\,. \tag{3}$$

This is a set of algebraic equations that can be solved for *any* valid choice of the new clock, or equivalently, the delay function $\varDelta(q, p, T)$. Therefore, once the choice of the clock T' is made, the canonical variables of the new reduced phase space (q', p') are fully determined. This removes the spurious clock transformations that differ only in the choice of new canonical variables while introducing the same clock. But there is more to be said about the above equations.

Upon passing to quantum theory, we promote the basic variables q and p to respective operators $\widehat{Q}$ and $\widehat{P}$ as well as the constants of motions H_J to operators $\widehat{H}_J = H_J(\widehat{Q}, \widehat{P}, T)$, where a fixed operator ordering is assumed. Now, upon a clock transformation $(q, p, T) \mapsto (q', p', T')$ and passing to quantum theory, one must find exactly the same operators $\widehat{Q}$, $\widehat{P}$, and $\widehat{H}_J$ except for now the operators $\widehat{Q}$ and $\widehat{P}$ that correspond to new phase-space variables (q', p'). On the other hand, the operators $\widehat{H}_J$ correspond to exactly the same constants of motion as their physical meaning does not depend on the choice of clock. These constants of motion are, from the point of view of the Hamiltonian constraint system that we have reduced, the Dirac observables. And here is the big point to be made: all clocks are quantized in exactly the same manner in the sense that they are given the same representation of the Dirac observables. It is impossible to demand more equivalence between different clocks at quantum level. Therefore, if one finds in quantum theory any discrepancies between two clocks, it is because the clocks are different but not because they were based on different quantum representations. In other words, any ambiguities must be the effects of clock.

3. Gravity waves in a quantum universe

The canonical model for gravity waves in a flat Friedmann universe [14]

$$H_{\mathrm{TOT}} = H_{\mathrm{FRW}} + H_{\mathrm{GW}}\,, \tag{4}$$

$$H_{\mathrm{FRW}} = p^2\,, \qquad H_{\mathrm{GW}} = -\frac{1}{2}\left|\pi_\pm\left(\vec{k}\right)\right|^2 - \frac{1}{2}\left(k^2 - \frac{\ddot{a}}{a}\right)\left|\mu_\pm\left(\vec{k}\right)\right|^2\,, \tag{5}$$

where the amplitude of the gravity wave reads $h_\pm(\vec{k}) = \frac{\mu_\pm(\vec{k})}{a}$. We easily identify the constants of motion both for the background and for the perturbation. Applying the set of algebraic equations (3) discussed above and choosing the delay function in the form $\Delta = \Delta(a,p)$, we find new variables $(a', p', \mu'_\pm, \pi'_\pm)$ and a new clock T'.

Upon passing to quantum theory, we obtain different quantum reduced phase spaces for different clocks. However, we may fix a reduced phase space, say, corresponding to some initial choice of clock and make comparisons between clocks by studying dynamical operators in this fixed reduced phase space. Furthermore, we may make a semi-classical approximation to the dynamics in the background phase space. In this method, we simply compare background trajectories in a fixed reduced phase space. In the left panel of Fig. 1, we find a semi-classical trajectory in the background reduced phase space obtained with some initial choice of clock. We see that the trajectory reverses close to the $a = 0$ singularity, which is the effect of a semi-classical $\hbar^2$-correction to the dynamics, and represents a universe that first contracts, then bounces, and finally re-expands. It is the bouncing part of the trajectory that arises due and is sensitive to the semi-classical correction and thus, to the effect of switching between the clocks as we now show.

In Fig. 1, we plot dynamical trajectories for a few clocks but in a unique reduced phase space (a,p). We observe how switching between clocks may alter the bouncing dynamics. In some clocks, the trajectories bounce earlier than in others. In some clocks, the trajectories make a single bounce,

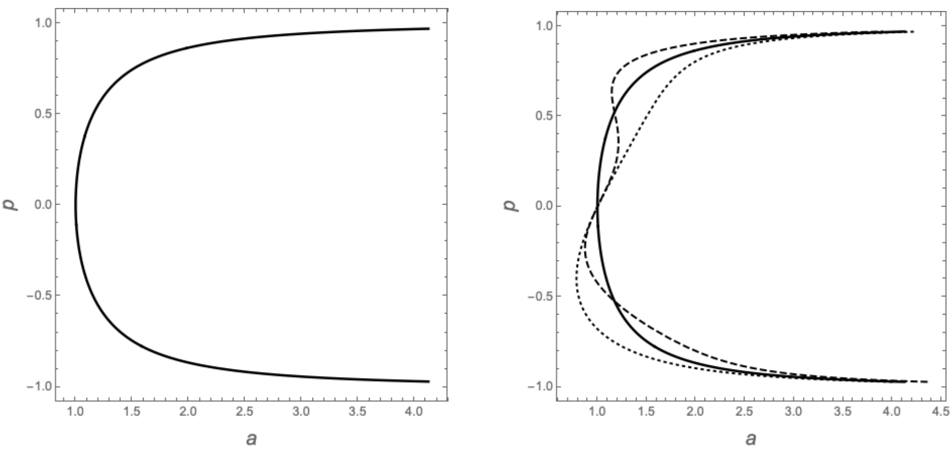

Fig. 1. Left: a dynamical trajectory of the background geometry undergoes a bounce from negative to positive ps generated by a semi-classical correction to the dynamics. Right: a set of dynamical trajectories produced in different clocks and mapped into a single reduced phase space (a, p).

while in others they make many bounces. In some clocks, the bounce is very symmetric, whereas in others, the expanding and contracting branches differ significantly. All these differences occur in the phase-space region where the trajectories are driven by the semi-classical correction. Once the trajectories move sufficiently far away from the boundary $a = 0$, they approach the respective classical trajectory irrespectively of the employed clock. Therefore, we can make a very important observation that the clock effect must vanish asymptotically for large and classical universes in which semi-classical corrections are negligible. Hence, despite the fact that there is not much physical that we can say about the bounce itself except for that it happened, we can make definite predictions about the asymptotic classical states of the cosmological system.

In Fig. 2, we plot dynamical trajectories for a few clocks but in a unique reduced phase space (a, h_k), where h_k is the amplitude of a gravity wave for wavevector k. As in the case of the background trajectories, the present trajectories may significantly differ close to the bounce, where the semi-classical correction is important. Once the universe sufficiently expands, the gravity-wave amplitudes converge for all clocks. Therefore, we confirm our previous observation that the clock effect must vanish asymptotically for large and classical universes in which semi-classical corrections are negligible.

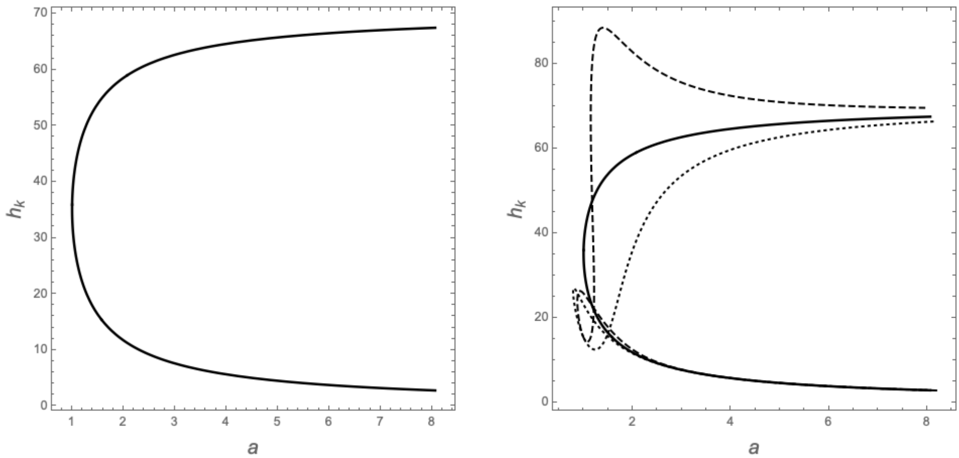

Fig. 2. Left: the gravity wave amplitude h_k *versus* the scale factor a. Initially, for large and decreasing a, the amplitude is small, then for small and decreasing a, the amplitude grows and continues to grow through the bounce until it becomes constant for large and growing a. Right: a set of dynamical trajectories produced in different clocks and mapped into a single reduced phase space (a, h_k).

4. Interpretation

Let us recapitulate all the relevant facts about clocks in quantum gravity: *(i)* Upon clock transformations, quantum constants of motion are invariant, whereas quantum dynamical variables and the evolution of their expectation values in general vary. Hence, clocks influence the dynamical content of quantum gravity; *(ii)* The expectation values of the dynamical variables are invariant with respect to clock transformations away from the bounce where the behavior of the expectation values becomes classical. This is thus a phase-space region in which an unambiguous prediction for the large universe can be made; *(iii)* Depending on the choice of clock and the choice of a dynamical variable, one may postpone the convergence of the expectation value of the latter to the respective classical trajectory until the universe is as large as one wishes. There is no threshold in the size of the universe beyond which the expectation values of all dynamical variables behave classically.

We note that on the one hand, we found that asymptotically for infinite universe, the dynamical predictions of quantum gravity do not depend on clock. This is a very strong result that we may rephrase as follows: despite that the dynamical variables are not Dirac observables from the point of view of the Hamiltonian constraint theory, they are able to provide unambiguous predictions for the large universe, which is exactly what we can observe and measure. On the other hand, for a given size of the universe and for a given clock, only some dynamical variables exhibit classical behavior. Equivalently, for any size of the universe and for any clock, there are dynamical variables that do not exhibit classical behavior. How to understand this inconsistency? We interpret it as the usual problem with semi-classical descriptions of systems such as the harmonic oscillator, in which only expectation values of the simplest observables exhibit classical behavior, while more complex observables diverge from classical behavior due to $\hbar^2$-corrections. Hence, this particular problem is not restricted to the gravitational systems and to the issue of time. We thus postulate to use only these dynamical variables whose expectation values behave classically in a clock in which predictions are made. A more detailed analysis of the above model and a more thorough discussion of the obtained results are presented in the forthcoming paper [14].

The author acknowledges the support of the National Science Centre, Poland (NCN) under the research grant 2018/30/E/ST2/00370.

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