

COMPLEX SCALAR FIELD
IN κ -MINKOWSKI SPACETIME*

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It is often expected that one cannot treat spacetime as a continuous manifold as the Planck scale is approached, because of possible effects due to a quantum theory of gravity. There have been several proposals to model such a deviation from the classical behaviour, one of which is the noncommutativity of spacetime coordinates. In this context, the non-commutativity scale is seen as an observer-independent length scale. Of course, such a scale imposes a modification of ordinary relativistic symmetries, which now need to be deformed to accommodate this fundamental scale. The κ -Poincaré algebra is an example of this deformation. In what follows, I will briefly describe a construction of a κ -deformed complex scalar field theory, while at the same time shedding light on the behaviour of discrete and continuous symmetries in this formalism. This in turn will open the way to the study of the application of this formalism to actual physical processes. I will then conclude with some comments and prospects for the future.

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1. From non-commutative spacetime to the action

We will present a brief preview of the works in Refs. [1, 2]. For a more detailed introduction to the framework and the formalism, see for example Ref. [3]. The starting point is κ -Minkowski spacetime, whose coordinates satisfy the $\mathfrak{an}(3)$ Lie algebra defined as $[\hat{x}^0, \hat{x}^i] = i\hat{x}^i/\kappa$. Notice that one recovers canonical Minkowski spacetime in the formal limit $\kappa \rightarrow \infty$. To have a more physical understanding of the above commutator, notice that $1/\kappa$ has dimensions of length, and it is sometimes identified with the Plank length. In order to build fields in this spacetime, we need to define plane waves first. To do so, one can proceed as follows. First, pick an explicit representation of the $\mathfrak{an}(3)$ Lie algebra in terms of matrices (it turns out

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that the lowest dimensional one has dimension 5). Then, one can define a plane wave as an element \hat{e}_k of the relative $AN(3)$ Lie group, *i.e.* for example¹ $\hat{e}_k = \exp(i\mathbf{k}_i \hat{\mathbf{x}}^i) \exp(ik_0 \hat{x}^0)$. Notice that, since the elements of the matrices \hat{x}^μ are dimensionful, so are the parameters k_μ . Written explicitly, one has

$$\hat{e}_k = \begin{pmatrix} \frac{\bar{p}_4}{\kappa} & \frac{\mathbf{k}}{\kappa} & \frac{p_0}{\kappa} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ \frac{\bar{p}_0}{\kappa} & -\frac{\mathbf{k}}{\kappa} & \frac{p_4}{\kappa} \end{pmatrix}, \quad (1)$$

where²

$$p_0 = \kappa \sinh \frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \quad \mathbf{p}_i = k_i e^{k_0/\kappa} \quad p_4 = \kappa \cosh \frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa}. \quad (2)$$

Both k_μ and $p_A(k)$ ($A = 0, 1, 2, 3, 4$) are two different coordinates of momentum space. Notice that the p_A are not independent, since $-p_0 + \mathbf{p}^2 + p_4^2 = \kappa^2$ (notice also that $p_+ = p_0 + p_4 > 0$). It is therefore clear that momentum space is curved. This is reflected in a deformed sum of momenta, which is defined through the group property $\hat{e}_k \hat{e}_l =: \hat{e}_{k \oplus l}$. At the same time, inverse momenta are defined in terms of group inverses $\hat{e}_k^{-1} =: \hat{e}_{S(k)}$, and $S(k)$ is called the antipode of k . In order to simplify the treatment of integrals and derivatives, we can use a³ Weyl map \mathcal{W} to send group elements \hat{e}_k into canonical plane waves $e_p := \exp(ip_\mu(k)x^\mu)$. The group law is preserved thanks to the \star product defined by $\mathcal{W}(\hat{e}_{k \oplus l}) = e_{p(k) \oplus q(l)} =: e_{p(k)} \star e_{q(l)}$. In general, the \star product is not commutative. Because of this, we choose the following action for a κ -deformed complex scalar field:

$$S = \frac{1}{2} \int_{\mathbb{R}^4} d^4x \left[(\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) + (\partial^\mu \phi) \star (\partial_\mu \phi)^\dagger - m^2 \left(\phi^\dagger \star \phi + \phi \star \phi^\dagger \right) \right]. \quad (3)$$

To obtain the equations of motion, one usually integrates by parts, but in this deformed context, derivatives do not satisfy the Leibniz rule. In fact, if they did, one could get the following contradiction:

$$i(p \oplus q)_\mu e_{p \oplus q} = \partial_\mu (e_p \star e_q) = (\partial_\mu e_p) \star e_q + e_p \star \partial_\mu e_q = i(p + q)_\mu e_{p \oplus q}. \quad (4)$$

¹ A different choice in the ordering results in a change of coordinates in momentum space.

² The explicit expression of \bar{p}_4 and \bar{p}_0 in terms of k_μ is not relevant to the present discussion.

³ There are several equivalent ways in which a suitable Weyl map can be chosen, see Ref. [1] for more details.

Hence, one has to use the appropriate deformations for the Leibniz rules. After some computations, one can verify that the equations of motion satisfied by the field are the canonical Klein–Gordon equations, and the field which satisfies them can be written as

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \xi(p) a_{\mathbf{p}} e^{-i(\omega_p t - \mathbf{p} \cdot \mathbf{x})} + \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \xi(p) b_{\mathbf{p}^*}^\dagger e^{i(S(\omega_p)t - S(\mathbf{p}) \cdot \mathbf{x})}. \quad (5)$$

The action of the discrete transformations C, P, T can be defined in the usual way in the deformed context, *i.e.* $T\phi(t, \mathbf{x})T^{-1} = \phi(-t, \mathbf{x})$, $P\phi(t, \mathbf{x})P^{-1} = \phi(t, -\mathbf{x})$, and $C\phi(t, \mathbf{x})C^{-1} = \phi^\dagger(t, \mathbf{x})$. Notice that this is due to the presence of the antipode in the on-shell field. Furthermore, the action is manifestly invariant under both CPT and κ -deformed Lorentz transformations.

2. Charges and features of the model

We now need to compute the charges for our model. There are two main ways to do so. The first is by direct computation using the Noether theorem. However, the fact that derivatives do not follow the Leibniz rule makes this job a prohibitively difficult one, except for the case of translation charges. The second way is to use the covariant phase-space formalism. Although very direct, this second method needs to be carefully defined in the deformed context. We chose a hybrid approach, namely we derived the translation charges from the Noether theorem, and then we built a covariant phase-space approach which was able to reproduce the translation charges, allowing then to compute the remaining charges. Given a symplectic form Ω , and a symmetry in spacetime described by the vector field ξ , the charge Q_ξ can be defined by $-\delta_\xi \lrcorner \Omega = \delta Q_\xi$. In this context, δ_ξ is a vector field in phase space describing the variation $\delta_\xi A$ of any physical quantity A in phase space under the action of the symmetry ξ in spacetime, δ is the exterior derivative in phase space, and \lrcorner represents a contraction. The translation charges computed directly from the Noether theorem are the following⁴:

$$\mathcal{P}_\mu = \int d^3p \alpha(p) \left[-S(p)_\mu a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + p_\mu b_{\mathbf{p}^*}^\dagger b_{\mathbf{p}^*} \right]. \quad (6)$$

The quantity $\alpha(p)$ is a function of momenta whose explicit expression does not concern us in this context. To reproduce Eq. (6) in the covariant phase-space formalism, we need to assume the following deformation of the canonical contraction rule between vector fields and forms⁵:

$$\delta_v \lrcorner (A \wedge B) = (\delta_v A)B + A(S(\delta_v)B), \quad (7)$$

⁴ There is also a fifth charge \mathcal{P}_4 , see [1] for further details.

⁵ Recall that the canonical relation is $\delta_v \lrcorner (A \wedge B) = (\delta_v A)B - A(\delta_v B)$.

where $S(\delta_v)$ can be appropriately defined [2]. Using this definition, one can compute all the remaining charges, and the creation/annihilation operators algebra (which turns out to be the canonical one). For example, the boost charge \mathcal{N}_i is

$$\mathcal{N}_i = -\frac{1}{2} \int d^3p \, \alpha \left\{ S(\omega_p) \left[\frac{\partial a_{\mathbf{p}}^\dagger}{\partial S(\mathbf{p})^i} a_{\mathbf{p}} - a_{\mathbf{p}}^\dagger \frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i} \right] + \omega_p \left[b_{\mathbf{p}} \frac{\partial b_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i} b_{\mathbf{p}}^\dagger \right] \right\}. \quad (8)$$

One can then verify that the charges satisfy the usual non-deformed Poincaré algebra. However, due to the effects of κ -deformation, one can also verify that, *e.g.*, $[\mathcal{N}_i, C] \neq 0$, meaning that CPT symmetry is subtly violated. More explicitly, using the definition

$$C = \int d^3p \, \left(b_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right), \quad (9)$$

one can show that $[\mathcal{N}_i, C]$ is given by

$$\begin{aligned} & \frac{i}{2} \int d^3p \left\{ S(\omega_p) \left[\frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i} b_{\mathbf{p}}^\dagger - a_{\mathbf{p}} \frac{\partial b_{\mathbf{p}}^\dagger}{\partial S(\mathbf{p})^i} + \frac{\partial a_{\mathbf{p}}^\dagger}{\partial S(\mathbf{p})^i} b_{\mathbf{p}} - a_{\mathbf{p}}^\dagger \frac{\partial b_{\mathbf{p}}}{\partial S(\mathbf{p})^i} \right] \right. \\ & \left. + \omega_p \left[\frac{\partial b_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i} a_{\mathbf{p}} - b_{\mathbf{p}}^\dagger \frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i} + \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i} a_{\mathbf{p}}^\dagger - b_{\mathbf{p}} \frac{\partial a_{\mathbf{p}}^\dagger}{\partial \mathbf{p}^i} \right] \right\}. \end{aligned} \quad (10)$$

Notice that in the limit $\kappa \rightarrow \infty$, one recovers the canonical result $[\mathcal{N}_i, C] = 0$.

3. Comments and conclusions

The fact is that $[\mathcal{N}_i, C] \neq 0$ has several important physical consequences. The most apparent one is a difference in the decay time between particles and antiparticles. Furthermore, we now have a well-defined theory which will allow us to study in detail the propagator and the n -point functions in general. From this point of view, it will be interesting to tackle the loops in this deformed context. Finally, what has been done for the case of the complex scalar field will be extended to fields of higher spins, in order to expand the discussion to more realistic phenomena.

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