STRUCTURE FORMING PLANE SYMMETRIC DUST INHOMOGENEOUS COSMOLOGICAL MODEL*

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We present a special case of the plane symmetric model from the G_3/S_2-symmetric space-times solving the Einstein equations for a dust source which exhibits a controlled form of the growth of finite matter density inhomogeneities.

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1. Introduction

We consider a model whose space-time belongs to the G_3/S_2-symmetric class of space-times and is plane symmetric. We assume that this space-time solves the Einstein equations without the cosmological constant for a dust source. This model is a planar counterpart of the Lemaître–Tolman model which is spherically symmetric. We will consider an infinite regular arrangement of inhomogeneities in the form of stacked planes of over- and underdensities. We are going to determine the dependence of the temporal evolution of the energy density contrast with regard to the specifics of the inhomogeneities. Such a formula could be useful for the modeling of the large-scale structure formation. There exist some formulas, which describe the evolution of the energy density contrast for general profile of inhomogeneities [1, 2], but they are limited to the cases with small contrast.

2. Definition and general properties of the model

The space-time metric and the matter four-velocity of the model are given in coordinates $t \in \mathbb{R}_+$, $x, y, z \in \mathbb{R}$ as follows:

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$$g_{mn} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & q^2 & 0 & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p^2 \end{pmatrix}, \qquad u_n = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{1}$$

where

$$q = \frac{p'}{o}, \qquad \dot{p}^2 = \frac{n}{p} + o^2 . \tag{2}$$

The quantity p is a function of coordinates t and x, quantities o and n are functions of x. The dot and prime denote the derivative with respect to the coordinate t and x respectively.

The matter in the model is defined as a dust fluid whose flow is geodesic and irrotational. The magnetic part of the Weyl tensor with respect to this flow also vanishes. The basic nonzero scalar quantities in the model are

— the expansion rate

$$\theta = \frac{\left(p^2 \dot{p}\right)'}{p^2 p'}, \tag{3}$$

— the Ricci scalar of the spatial hypersurfaces

$${}^3R = -\frac{2\left(o^2 p\right)'}{p^2 p'}, \tag{4}$$

— the energy density

$$\varkappa\rho = \frac{n'}{p^2 p'}, \tag{5}$$

where $\varkappa = \frac{8\pi G}{c^4}$, where c is the speed of light and G is the gravitational constant.

The equation for the function p can be integrated parametrically to give the solution in the form

$$p = \frac{n}{2o^2}(\cosh\eta - 1), \qquad \frac{n}{2o^3}(\sinh\eta - \eta) = t - m, \tag{6}$$

where m is an arbitrary function of x. For practical reasons, the derivative of the function p with respect to the coordinate x is very useful and it can be given as [3]

$$p' = \left(\frac{n'}{n} - \frac{\left(o^2\right)'}{o^2}\right) p - \left(\left(\frac{n'}{n} - \frac{3\left(o^2\right)'}{2o^2}\right)(t - m) + m'\right) \dot{p} . \tag{7}$$

The model is fully determined by the three functions of the x coordinate, m, n, and o. However, since we have freedom to choose the x coordinate, effectively, we have only two free functions to restrict physical properties of the model.

The homogeneous limit of the considered model can be achieved by the following choice:

$$m' = 0\,, \qquad n = \frac{\Omega_M}{(1-\Omega_M)^{\frac{3}{2}}}\,o^3\,. \tag{8}$$

Then, it is the spatially open Friedmann–Lemaître model characterized by the energy density parameter $\Omega_M \in (0,1)$. When only $n \propto o^3$, then the model becomes asymptotically homogeneous in the infinite future. Such a model allows inhomogeneities only to decay. When only $m' = 0$, then the model becomes asymptotically homogeneous in the past, near the initial singularity. Such a model allows inhomogeneities only to grow.

The inhomogeneous nature of the considered model manifests in the asymptotic behavior of the Ricci scalar of the spatial hypersurfaces and the energy density. The asymptotic profile of the Ricci scalar of the spatial hypersurfaces in the past depends on the x coordinate and equals, when $m \neq 0$,

$$\lim_{t\to 0} {}^3R = -\frac{2\left(\frac{2}{3}\right)^{\frac{4}{3}} o^2}{n^{\frac{2}{3}}}(t-m)^{-\frac{4}{3}}\,, \tag{9}$$

and when $m = 0$,

$$\lim_{t\to 0} {}^3R = -\frac{2\left(\frac{2}{3}\right)^{\frac{4}{3}}}{n^{\frac{2}{3}}}\left(o^2 + \frac{3n\left(o^2\right)'}{n'}\right) t^{-\frac{4}{3}}\,. \tag{10}$$

Similarly, the asymptotic profile of the energy density in the future depends on the x coordinate and equals [4]

$$\lim_{t\to\infty} \varkappa\rho = \frac{n'}{o^2 o'}(t-m)^{-3}\,. \tag{11}$$

As it was noted in [5] in the context of the theory of cosmological perturbations, decaying and growing modes of density perturbations are of different nature since the growing mode is present only if curvature perturbations of spatial hypersurfaces are nonzero. In evolution equations, curvature perturbations are the source term for the growing density perturbations mode. Similar result holds for the hyperbolic Lemaître–Tolman model [6, 7] where, however, density inhomogeneities do not grow unboundedly, but their contrast saturates to a finite value. In our model in the same manner, final inhomogeneities in the energy density develop due to initial inhomogeneities in the Ricci scalar of the spatial hypersurfaces.

3. Details and specific properties of the model

We are going to construct a model which comprises only growing inhomogeneities, so we choose

$$m = 0\,. \tag{12}$$

This also means that the initial singularity is simultaneous in the whole space-time. Furthermore, we want the model not to exhibit shell crossings and to be always expanding. This could be achieved with the general assumption that $n, o > 0$ and, in particular, we assume

$$o = \exp x\,. \tag{13}$$

In such a model, we do not expect the structure to virialize or to collapse, eventually. During the evolution, the energy density is decreasing everywhere, but its profile becomes frozen and stands into infinite future.

The homogeneous limit of the asymptotic profile of the energy density in the future reads $3\Omega_M(1-\Omega_M)^{-\frac{3}{2}}t^{-3}$. Thus, we choose to define the function n as a solution to the following asymptotic profile of the energy density in the future:

$$\lim_{t\to\infty} \varkappa\rho = \frac{3\mu}{(1-\mu)^{\frac{3}{2}}}t^{-3}\left(1+\kappa\cos\left(\frac{x}{\lambda}\right)^{2\nu}\right), \tag{14}$$

where, for the constant of integration, we assume that $\lim_{x\to-\infty} n = 0$. We have introduced here four parameters κ, λ, μ, and ν which determine the properties of inhomogeneities. The assumed asymptotic profile of the energy density in the future has a form of an infinite regular chain of identical planar overdensities having a cosine-like shape.

The parameter μ is restricted to $\mu \in (0,1)$. When there is no inhomogeneities, it plays the role of the energy density parameter Ω_M. It controls the time of the structure formation, which is earlier for smaller values of μ. The parameter κ takes values $\kappa \in [0,\infty)$. It equals to the asymptotic value of the energy density contrast in the future. The parameter ν is considered to be a natural number, $\nu \in \mathbb{N}_+$. It controls the final width of the inhomogeneities. The parameter λ, $\lambda \in \mathbb{R}_+$, determines the distribution of the inhomogeneities. For example, for $\lambda = 10^{-2}$, the observer will count about 10 overdensities up to the redshift of about 10. The parameter λ is a natural small parameter in the model and thus we will consider only cases with $\lambda \ll 1$. In particular, for such small values of λ, the function η very weakly depends on the x coordinate and then its formula reads

$$\frac{1}{2}(1+f\kappa)\frac{\mu}{(1-\mu)^{\frac{3}{2}}}(\sinh\eta-\eta) = t\,, \qquad f = \frac{(2\nu)!}{2^{2\nu}(\nu!)^2}\,. \tag{15}$$

We define the energy density contrast as a contrast between a local maximum and minimum of the energy density

$$\delta = \frac{\rho_{\mathrm{max}} - \rho_{\mathrm{min}}}{\rho_{\mathrm{min}}} \,. \tag{16}$$

Usually, the energy density contrast is defined with respect to the averaged density which requires the adoption of some averaging procedure. Since there are many averaging formalisms and there is no consensus on which one is correct [8, 9], we decided to define the energy density contrast without referring to averaging. From this definition, we may find

$$\delta = (1+\kappa)\frac{1+f\kappa g}{1+\kappa-(1-f)\kappa g} - 1 \,, \tag{17}$$

where

$$g = 1 + 3\frac{1-\frac{\eta}{2}\coth\left(\frac{\eta}{2}\right)}{\sinh^2\left(\frac{\eta}{2}\right)} \,. \tag{18}$$

We can see that the energy density contrast directly depends on the parameters κ and ν.

4. Summary

We have developed a simple inhomogeneous cosmological model for which we have determined the energy density contrast using a definition that does not require averaging. The proposed definition is suitable for describing the contrast for evenly distributed identical inhomogeneities. According to the obtained formula, the temporal evolution of the energy density contrast explicitly depends on the properties of the inhomogeneities, in particular, on the asymptotic value of the energy density contrast in the future. This is significant because the averaging-dependent approaches present in the literature (*e.g.* [2]) suggest that the evolution of the energy density contrast is universal and depends on the details of the inhomogeneities only implicitly. This result highlights the importance of inhomogeneity details in describing the evolution of cosmological models, especially in the context of large-scale structure formation studies and in the interpretation of cosmological data. Along with the growth rate of inhomogeneities, the asymptotic saturation level of their energy density contrast should be considered as the basic parameter characterizing the cosmological model.

REFERENCES

[1] S.W. Goode, J. Wainwright, *Phys. Rev. D* **26**, 3315 (1982).
[2] M. Kasai, *Phys. Rev. D* **47**, 3214 (1993).
[3] A. Barnes, *J. Phys. A: Math. Gen.* **3**, 653 (1970).
[4] C. Hellaby, A. Krasiński, *Phys. Rev. D* **73**, 023518 (2006).
[5] S. Sikora, K. Głód, *Eur. Phys. J. C* **81**, 208 (2021).
[6] J. Wainwright, S. Andrews, *Class Quantum Grav.* **26**, 085017 (2009).
[7] R.A. Sussman, *Class. Quantum Grav.* **30**, 235001 (2013).
[8] R.J. van den Hoogen, *J. Math. Phys.* **58**, 122501 (2017).
[9] A.A. Coley, G.F.R. Ellis, *Class. Quantum Grav.* **37**, 013001 (2020).