BEYOND GEOMETRIC OPTICS LIMIT OF GRAVITATIONAL WAVE LENSING IN PALATINI $f(\hat{R})$ GRAVITY*

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The planned next generation of detectors such as Einstein Telescope, Cosmic Explorer, and space-based detectors such as LISA are likely detect gravitational waves signals more frequently than the current generation LIGO detectors. With such an increased frequency of detection, we expect some of the signals to be gravitationally lensed. An opportunity that lensing opens up is to test different theories of gravity. In this work, we study gravitational lensing in the context of Palatini $f(\hat{R})$ gravity using the WKB approximation in the geometric optics limit and beyond.

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1. Introduction

The phenomenon of bending of electromagnetic wave by massive objects such as the Sun has been considered as one of the fundamental tests of General Relativity (GR). The famous eclipse expedition by A. Edington and team in 1919 and subsequent observations have made Einstein and his theory overnight famous. Predictions of GR were far ahead for the technology available in the next few decades. However, physicists predicted the possibility of massive objects to act like a geometric lens and produce multiple images. Beyond being a spectacular view Zwicky [1, 2] realised that galaxies could act as natural cosmic telescopes to view faint background sources. Refsdal's proposal [3] to use time delay between multiple images to estimate Hubble constant has revived interest in lensing. The advancements in technology have made it possible to observe several lensed events and now the field of gravitational lensing has matured enough to become an unavoidable tool in astronomy [4].

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One of the most important scientific achievements in this century is the direct detection of gravitational waves by the LIGO Science Collaboration [5]. This new window of gravitational wave astronomy opens up several possibilities. One of them being the chances of detecting a lensed gravitational signal. Like electromagnetic waves, gravitational waves can also be lensed by the matter distribution between the source and the detector. However, unlike electromagnetic waves where we employ geometric optics limit, GWs need to be dealt with in two different regimes: Geometric Optics (GO) and Wave Optics (WO) limit. This is due to the larger wavelength of GW signals ($\lambda_{\rm gw} \sim 10^{10}$–$10\,{\rm km}$), in comparison to the typical scale of electromagnetic waves ($\lambda_{\rm em} \sim 10^{-8}$–$10^{-2}\,{\rm m}$). For the frequency window in which LIGO operates, one can safely apply the concepts of GO. However, for the next generation ground-based detectors such as the Einstein Telescope(ET) and Cosmic Explorer (CE) as well as the planned space based observatories such as LISA and DECIGO, the WO effects become more important.

The main observational features of GW lensing are the frequency dependent amplification and multiple gravitational signals [6, 7] of the same source due to different geometric paths around the gravitational lens. The time delay between the signals depends on the type of lens and the chance alignment between the source and the lens. Like in the usual lensing, the signals from far away source are amplified and hence their Signal to Noise Ratio (SNR) is increased. Lensing of gravitational waves offers some interesting applications like estimation of cosmological parameters from GW beat pattern [8]. Another opportunity that such a lensing phenomenon opens up is that it can be used to test modified theories of gravity. In this article, we take $f(\hat{R})$ gravity in Palatini formalism and study the propagation of gravitational waves in the lens background. The article has been organised as follows; In Section 2, we briefly explain the main features of Palatini $f(\hat{R})$ gravity. The predictions of GO limit and beyond GO limit have been explained in Section 3, and we conclude with our results.

2. Palatini $f(\hat{R})$ gravity

In this section, we briefly introduce the Palatini $f(\hat{R})$ gravity for the readers convenience. In the Palatini formalism, we treat the metric and the connection to be independent quantities. Therefore, the Riemann tensor does not depend on the metric, we denote this difference by a $(\hat{.})$ over the symbols. For example, we write the Riemann tensor as $\hat{R}_{\mu\nu}$ and the Palatini Ricci scalar as $\hat{R} = g^{\mu\nu}\hat{R}_{\mu\nu}(\mathit{\Gamma})$, where $\mathit{\Gamma}$ is the independent connection. The action in Palatini $f(\hat{R})$ gravity can be written as

$$S[g,\mathit{\Gamma},\psi_m] = \frac{1}{2\kappa}\int \sqrt{-g}f\left(\hat{R}\right){\rm d}^4x + S_m[g,\psi_m]\,, \tag{1}$$

where S_m is the matter action which depends only on the metric and the matter fields ψ_m and $\kappa = -8\pi G/c^4$. As this action is a function of two dynamical variables we obtain two different equations of motion. Now varying this action with respect to the metric gives

$$f'\left(\hat{R}\right)\hat{R}_{\mu\nu} - \frac{1}{2}\left(\hat{R}\right)g_{\mu\nu} = \kappa T_{\mu\nu}\,, \tag{2}$$

where $f'(\hat{R}) = \frac{\mathrm{d}f}{\mathrm{d}\hat{R}}$, a differentiation w.r.t. the curvature and $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g_{\mu\nu}}$ is the energy momentum tensor. Taking a trace of the above field equation with the metric $g_{\mu\nu}$ gives a structural equation, which is an algebraic relation between the Palatini Ricci scalar and the trace of the energy momentum tensor ($T = g^{\mu\nu}T_{\mu\nu}$)

$$f'\left(\hat{R}\right)\hat{R} - 2f\left(\hat{R}\right) = \kappa T\,. \tag{3}$$

Similarly, variation of Eq. (1) with respect to the independent connection $\Gamma^{\alpha}_{\mu\nu}$ gives

$$\hat{\nabla}_{\beta}(\sqrt{-g}f'\left(\hat{R}(T)\right)g^{\mu\nu}) = 0\,, \tag{4}$$

where $\hat{\nabla}_{\beta}$ indicates that the covariant derivative is taken with respect to the independent connection Γ and also we assume the connection to be symmetric. The above equation can be simplified by defining a new metric which is conformally related, $\hat{g}_{\mu\nu} = f' g_{\mu\nu}$. Using this, the above equation can be rewritten as

$$\hat{\nabla}_{\beta}\left(\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\right) = 0\,. \tag{5}$$

Therefore, the independent connection Γ turns out to be a Levi-Civita-like connection for the new metric $\hat{g}_{\mu\nu}$ and it reduces to the Levi-Civita connection for the metric $g_{\mu\nu}$ if the conformal factor $f'(\hat{R})$ becomes a constant. The metric approach to $f(R)$ and the Palatini approach become equivalent only in the case where $f(\hat{R}) = R$, where both reduce to GR. There exists a scalar tensor representation of field equations to Palatini $f(\hat{R})$ gravity. However, the scalar field here is non-dynamical. For more details see [9]. The field equation becomes

$$\hat{G}_{\mu\nu} = \kappa\hat{T}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{U}\left(f'\right)\,, \tag{6}$$

where $\hat{G}_{\mu\nu}$ is the Einstein tensor written in terms of conformal variables while $\hat{T}_{\mu\nu} = (f')^{-1}T_{\mu\nu}$. The potential $\hat{U}(f')$ carries the information on the form of f' and in the case of the vanishing energy-momentum tensor trace, $U(f')$ can be neglected as it is identified with the cosmological constant.

3. Propagation of gravitational waves in beyond geometric optics limit

To study the propagation of gravitational waves in this theory, we need to perturb the field equations. It is more convenient to work in the scalar tensor representation of Palatini $f(\hat{R})$ gravity which is Eq. (6) and using the following perturbation

$$\hat{g}_{\mu\nu} = \overset{\mathrm{B}}{\hat{g}}_{\mu\nu} + \hat{h}_{\mu\nu}\,, \tag{7}$$

where $\overset{\mathrm{B}}{\hat{g}}_{\mu\nu} = f'\overset{\mathrm{B}}{g}_{\mu\nu}$ is the slowly varying background created by gravitational lens and $\hat{h}_{\mu\nu} = f' h_{\mu\nu}$ is the gravitational wave perturbation. However, for the particular case of anisotropic-free stress, the perturbations $\hat{h}_{\mu\nu} = h_{\mu\nu}$ [11–13]. Under this assumption, we obtain the gravitational wave equation in curved spacetime as

$$\hat{\nabla}_\alpha \hat{\nabla}^\alpha h_{\mu\nu} - 2\hat{R}^{\tau}_{\rho\mu\nu} h^{\rho}_{\tau} = 0\,. \tag{8}$$

The propagation of gravitational waves in this lens background can be studied by using the Wentzel–Kramers–Brillouin (WKB) approximation[1]

$$h_{\mu\nu} = \mathrm{Re}\left\{\left[\xi^{(0)}_{\mu\nu} + \epsilon\xi^{(1)}_{\mu\nu} + \epsilon^2\xi^{(2)}_{\mu\nu} + \ldots\right]\mathrm{e}^{i\Phi/\epsilon}\right\}, \tag{9}$$

where ϵ in the above ansatz is a bookkeeping parameter to keep track on the order of expansion, and the scalar field $\Phi(x)$ defines the phase of the gravitational waves. The condition $\epsilon \longrightarrow 0$ defines the geometric optics limit. Substituting the eikonal ansatz (9) in the GW equation (8) we obtain the following eikonal expansion:

$$\begin{aligned}\hat{\nabla}^\alpha\hat{\nabla}_\alpha h_{\mu\nu} \quad & -2h_{\alpha\beta}\hat{R}^{\alpha}{}_{\mu\nu}{}^{\beta} = \mathrm{e}^{i\Phi/\epsilon}\left\{\frac{1}{\epsilon^2}\left[-\hat{k}^\beta\hat{k}_\beta\xi^{(0)}_{\mu\nu}\right]\right. \\ & +\frac{1}{\epsilon}\left[i\left(\hat{\nabla}_\beta\hat{k}^\beta\xi^{(0)}_{\mu\nu} + \hat{k}^\beta\hat{\nabla}_\beta\xi^{(0)}_{\mu\nu}\right) - \hat{k}^\beta\hat{k}_\beta\xi^{(1)}_{\mu\nu}\right] \\ & +\epsilon^0\left[\hat{\nabla}_\beta\hat{\nabla}^\beta\xi^{(0)}_{\mu\nu} + i\left[\hat{\nabla}_\beta\hat{k}^\beta\xi^{(1)}_{\mu\nu} + \hat{k}^\beta\hat{\nabla}_\beta\xi^{(1)}_{\mu\nu}\right.\right. \\ & \left.\left.\left. + \hat{k}^\beta\hat{\nabla}_\beta\xi^{(1)}_{\mu\nu}\right]\right]\right\} - 2h_{\alpha\beta}\hat{R}^{\alpha}{}_{\mu\nu}{}^{\beta} = 0\,,\end{aligned} \tag{10}$$

where the wave vector k^μ is defined as the gradient of the phase function $\hat{k}^\mu = \overset{\mathrm{B}}{\hat{g}}{}^{\mu\nu}\partial_\nu\Phi(x)$ and the covariant derivative $\hat{\nabla}$ is defined with respect to

[1] Also known as the eikonal approximation or stationary phase approximation in the literature.

the independent connection. Applying the eikonal ansatz (9) to the Hilbert gauge condition $\hat{\nabla}^\mu h_{\mu\nu} = 0$, we obtain

$$\mathrm{e}^{i\Phi/\epsilon}\left\{\frac{1}{\epsilon}\left[i\hat{k}^\mu \xi^{(0)}_{\mu\nu}\right] + \frac{1}{\epsilon^0}\left[i\hat{k}^\mu \xi^{(1)}_{\mu\nu} + \hat{\nabla}^\mu \xi^{(0)}_{\mu\nu}\right] + \mathcal{O}(\epsilon)\right\} = 0\,. \tag{11}$$

3.1. Geometric optics limit

The leading order term $\frac{1}{\epsilon^2}$ and the next-to-leading-order term $\frac{1}{\epsilon}$ in Eq. (10) defines the geometric optics limit of gravitational wave lensing. Analysing these terms gives us some of the important information regarding GW propagation. Starting from the leading term $\mathcal{O}(\epsilon^{-1})$ in the gauge condition, we see that the polarization tensor remains transverse to the propagation vector

$$\hat{k}^\mu \xi^{(0)}_{\mu\nu} = 0\,. \tag{12}$$

The leading order term $\mathcal{O}(\epsilon^{-2})$ in Eq. (10) gives the information that the wave vector is a null vector $\hat{k}^\mu k_\mu = 0$ and hence it travels at the speed of light. Taking the covariant derivative $\hat{\nabla}$ and making use of the fact that the wave vector $\hat{k}^\mu$ is the gradient of the phase function gives us the following relation

$$\hat{k}^\mu \hat{\nabla}_\mu \hat{k}_\nu = 0\,. \tag{13}$$

By the defining the wave vector as $\hat{k}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\hat{\lambda}}$ we can rewrite the above equation in a more familiar form

$$\frac{\mathrm{d}x^\beta}{\mathrm{d}\hat{\lambda}^2} + \hat{\Gamma}^\beta_{\alpha\mu}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\hat{\lambda}}\frac{\mathrm{d}x^\mu}{\mathrm{d}\hat{\lambda}} = 0\,, \tag{14}$$

where $\hat{\lambda}$ is the affine parameter in the conformal frame and $\hat{\Gamma}^\beta_{\alpha\mu}$ is the independent connection. The above equation gives us the information that GWs follow *autoparallel curves*, however using the expression for the connection and the affine parameter in terms of the physical metric $g_{\mu\nu}$, it can be shown that they coincide with the geodesics. The next-to-leading-order ($\mathcal{O}(\epsilon^{-1})$) contribution gives

$$2k^\alpha \hat{\nabla}_\alpha \xi^{(0)}_{\mu\nu} + \hat{\nabla}_\alpha \hat{k}^\alpha \xi^{(0)}_{\mu\nu} = 0\,. \tag{15}$$

We note that we can separate the wave tensor into the amplitude and polarization part as $\xi^{(0)}_{\mu\nu} = \mathcal{A}\mathcal{A}_{\mu\nu}$, where $\mathcal{A}$ is the amplitude, defined as $\mathcal{A} = \sqrt{\xi^*_{\mu\nu}\xi^{*\mu\nu}}$, and $\mathcal{A}_{\mu\nu}$ is the normalized polarization tensor. Using this decomposition in (15) along with the gauge condition we obtain two important pieces of information regarding the amplitude and the GW polarization tensor. The polarization tensor $\mathcal{A}_{\mu\nu}$ is parallel propagated along the propagation direction $\hat{k}^\mu$

$$\hat{k}^\alpha \hat{\nabla}_\alpha \mathcal{A}_{\mu\nu} = 0\,. \tag{16}$$

The amplitude equation can be rewritten by defining the momentum of the gravitons as $\hat{P} = \hbar \hat{k}^\mu$, and we obtain

$$\hat{\nabla}_\mu \hat{N}^\mu = 0\,, \tag{17}$$

where $\hat{N}^\mu = \frac{\mathcal{A}^2}{\hbar^2} \hat{P}^\mu$ is the graviton number density. Equation (17) implies the conservation of graviton number density in the ray bundle. However, both the conditions in (16) and (17) remain true only if the GW propagates along the wave vector $\hat{k}^\mu$ instead of k^μ. If the gravitational wave follows the geodesics of GR governed by the Levi-Civita connection, they are violated. The evolution of GW is known to depend on the background geometry which is studied in detail in [10].

3.2. Beyond geometric optics limit

Next, we take into account the beyond geometric optics corrections to GW propagation. The subleading term of the order $\mathcal{O}(\epsilon^0)$ in (10) and the corresponding term in the gauge condition (11) gives, respectively,

$$\begin{aligned} \hat{\nabla}_\beta \hat{\nabla}^\beta \xi^{(0)}_{\mu\nu} + i\left[\hat{\nabla}_\beta \hat{k}^\beta \xi^{(1)}_{\mu\nu} + 2\hat{k}^\beta \hat{\nabla}_\beta \xi^{(1)}_{\mu\nu}\right] - 2\xi^{(0)}_{\alpha\beta} \hat{R}^\alpha{}_{\mu\nu}{}^\beta &= 0\,, & (18)\\ i\hat{k}^\mu \xi^{(1)}_{\mu\nu} + \hat{\nabla}^\mu \xi^{(0)}_{\mu\nu} &= 0\,. & (19) \end{aligned}$$

The above equation can be rewritten in a compact form by introducing source tensors $S^{(0)}_{\mu\nu} = -i[2\xi^{(0)}_{\alpha\beta} \hat{R}^\alpha{}_{\mu\nu}{}^\beta - \hat{\nabla}_\beta \hat{\nabla}^\beta \xi^{(0)}_{\mu\nu}]$ and for the gauge condition $S^{g(0)}_\nu = i\hat{\nabla}^\mu \xi^{(0)}_{\mu\nu}$. The higher order terms in the expansion are sourced by the lower order terms

$$\hat{k}^\mu \xi^{(1)}_{\mu\nu} = S^{g(0)}\,. \tag{20}$$

Therefore, in the beyond geometric optics limit, the polarization tensor is not transverse to the propagation vector $\hat{k}^\mu$ and

$$\hat{\nabla}_\beta \hat{k}^\beta \xi^{(1)}_{\mu\nu} + 2\hat{k}^\beta \hat{\nabla}_\beta \xi^{(1)}_{\mu\nu} = S^{(0)}_\nu\,. \tag{21}$$

The presence of the source tensor will cause the polarization tensor to be smeared and would lead to the rise of additional polarizations. This is not particular to the Palatini $f(\hat{R})$ gravity but also reported in the beyond geometric limit of GR [14, 15]. However due to the presence of a conformal factor $f'(T)$, the propagation of GW and the predictions are different from GR.

4. Conclusion

The geometric optics and beyond geometric optics limit corrections to the propagation of GWs in the Palatini $f(\hat{R})$ gravity have been studied. In the geometric optics limit, we see that gravitational waves follow *autoparallel curves* which coincides with the geodesics. It has been shown that in the beyond GO limit, gravitational wave polarizations are not transverse to the propagation vector $\hat{k}^\mu$ and are sourced by lower order terms. Due to the presence of a source term, non-tensorial polarizations are expected to arise in the beyond GO limit which are unphysical and arise as a consequence of lensing. It has been shown in [14] that there does not exist a preferred class of observer which would measure only tensor polarizations, however, the vector polarizations can be made to vanish by doing a Lorentz transformation to a frame which is at rest with respect to the lens. A similar analysis is yet to be carried out for the Palatini $f(\hat{R})$ gravity, which will be the subject of a future work.

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