THE GENERAL RELATIVISTIC TWO-BODY PROBLEM AT 5PN*

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The general relativistic conservative compact binary dynamics is given through the fifth post-Newtonian (5PN) order. Through the 4PN order, the well-established methods and results get summarized. At the 5PN order, a recently completed computation is presented including comparisons with the literature. Three rational numbers are still under discussion. Terms not yet calculated at the 6PN order get pointed out.

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1. Introduction

The post-Newtonian (PN) approximation to General Relativity (GR) has proved very useful in applications, as well in the celestial mechanics including binary pulsar research as in the gravitational wave astronomy. The history of the PN approximation started in 1916 by Johannes Droste with 1PN but showing full control of the 1PN n-body dynamics only in 1938 thanks to the works by Eddington and Clark, and by Einstein, Infeld, and Hoffmann see, *e.g.* Ref. [1] for detailed history. The PN research culminated in 2014 with the publication of the novel nonlocal-in-time 4PN dynamics for compact binaries (particularly binary black holes) by Damour, Jaranowski, and Schäfer [2] based on the ADM canonical formulation of GR developed by Arnowitt, Deser, and Misner around 1960. Various confirmations of that 4PN dynamics were achieved in the years 2018–2020, based on a multipolar post-Newtonian ansatz within the Fokker-action approach by Blanchet and collaborators [3], and the effective field theory (EFT) approach by Foffa *et al.* [4] and Blümlein *et al.* [5]. Dimensional regularization has been applied using point particles as sources for the gravitational field represented by Dirac delta functions in d-dimensional position space. Apart

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from the ADM approach with its canonical coordinates, all the calculations made use of harmonic coordinates. The 5PN binary dynamics, performed in the years 2019–2022, has been achieved for the most part by the so-called Tutti-Frutti (TF) approach by Bini, Damour, and Geralico [6, 7], and in full by the EFT approach by Blümlein *et al.* [8, 9] with both approaches being based on bound- and scattering-state calculations. However, the complete agreement has not yet been achieved [10], including also results from EFT calculations by Foffa, Sturani, and Almeida [11, 12]. Results through 6PN order are also available, mainly based on the TF approach [7, 13].

2. The applied methods

The PN-approximation approach is defined by expanding the Einstein field equations and the equations of motion in a series of powers of $1/c^2$, where the order nPN means (c: velocity of light, G: Newtonian gravitational constant, M: total mass, r and v: typical radial separation and typical relative velocity in binaries): $\left(\frac{1}{c^2}\right)^n \sim \left(\frac{GM}{rc^2}\right)^l \left(\frac{v^2}{c^2}\right)^m$, $n = l + m$. Obviously, the PN approach is based on the virial theorem. The fast-motion-based post-Minkowskian (PM)-approximation approach is given as power series in terms of $G^n \sim \left(\frac{GM}{rc^2}\right)^n$. Hereof, by applying a slow-motion expansion, the PN approach can be recovered.

The following methods have played crucial roles: perturbation series (PN, PM, Multipolar, gravitational Self Force (SF)), bound- and scattering-state calculations, point particles as Black Holes (BHs) [extended bodies through 2PN only], dimensional regularization [throughout analytical regularization sufficient through 2PN only], canonical ADM formalism through 4PN, Fokker-action formalism through 4PN, EFT approaches (in–out and in–in), effective-one-body (EOB) theory, Delaunay averaging. From 5PN on, the recently developed TF approach, an efficient combination of most of the mentioned methods, has taken lead.

3. Analytical representation of binary black holes (BBHs)

3.1. Isolated BHs

In isotropic coordinates, the 3-space metric of an isolated BH reads

$$\mathrm{d}s^2 = \left(1 + \frac{GM}{2rc^2}\right)^4 \delta_{ij}\mathrm{d}x^i\,\mathrm{d}x^j = \left(1 + \frac{GM}{2r'c^2}\right)^4 \delta_{ij}\mathrm{d}x'^i\,\mathrm{d}x'^j\,,$$

where the inversion symmetry of the Einstein–Rosen bridge at event horizon is shown: $r'r = \left(\frac{GM}{2c^2}\right)^2$, $r'^2 = x'^i x'^i$, $r^2 = x^i x^i$.

3.2. Brill–Lindquist BBH metric: a velocity zero initial-value metric

The BBH initial-value 3-metric by Brill and Lindquist (1963) reads $\mathrm{d}s^2 = \left(1+\frac{1}{8}\phi\right)^4 \mathrm{d}\boldsymbol{x}^2$, $\phi = \frac{4G}{c^2}\left(\frac{\alpha_1}{r_1}+\frac{\alpha_2}{r_2}\right)$, $r_a = |\boldsymbol{x}-\boldsymbol{x}_a| \neq 0$, $(a=1,2)$, where ϕ is a homogeneous solution of $\Delta\phi = 0$ in punctured 3-space.

Surface integrals over the three asymptotic regions of the BBH punctured-space solution, enclosing the throats (event horizons), result in the single proper masses m_a as well as in the total mass (sum of proper masses plus binding mass–energy) in the form of the total Hamiltonian $H_{\mathrm{BL}} = (\alpha_1+\alpha_2)c^2$.

3.2.1. Spatial metric in *d*-dimensional space

Leaving punctured space for introducing matter sources, the application of d-dimensional space is necessary. The metric then reads

$$\begin{aligned} g_{ij} &= \left(1+\frac{d-2}{4(d-1)}\phi\right)^{\frac{4}{d-2}} \delta_{ij}\,, \\ \phi &= \frac{4\pi G^{(d)}}{c^2}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d}{2}}}\left(\frac{\alpha_1}{r_1^{d-2}}+\frac{\alpha_2}{r_2^{d-2}}\right). \end{aligned}$$

3.2.2. Point masses as sources for *d*-dimensional metric

The potential function ϕ results from d-dimensional delta-functions as sources

$$\delta^{(d)} = -\frac{\Gamma((d-2)/2)}{4\pi^{d/2}}\Delta^{(d)}\frac{1}{r^{d-2}}\,.$$

The formal positions of the point masses m_a are located in Euclidean d-coordinates space with metric δ_{ij}, obtained via conformal transformation of the d-metric g_{ij}, with densities $m_a\delta_a^{(d)} = m_a\delta^{(d)}(\boldsymbol{x}-\boldsymbol{x}_a)$, $\int m_a\delta_a^{(d)}\,\mathrm{d}^d x = m_a$. Notice that the support of the delta functions is not where the singularities of the true sources are located. It is rather like the virtual image charges in the electrostatics which are located outside the physical domain. The Hamiltonian-constraint part of the Einstein field equations reads

$$-\left(1+\frac{d-2}{4(d-1)}\,\phi\right)\Delta\phi = \frac{16\pi G^{(d)}}{c^2}\sum_a m_a\delta_a^{(d)}\,.$$

This equation is well defined if $1 < d < 2$ holds, allowing for dimensional regularization in the form of $(b \neq a)$,

$$\left(1+\frac{G^{(d)}(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}}\frac{\alpha_b}{r_{ab}^{d-2}}\right)\alpha_a\delta_a^{(d)} = m_a\delta_a^{(d)}\,.$$

4. Conservative BBH dynamics through 4PN

4.1. Near-zone and tail-field results

Using dimensional regularization for ultraviolet divergencies and analytical regularization for infrared ones, the near-zone conservative Hamiltonian and the time-symmetric, nonlocal-in-time, part one from backscattered waves in the far zone read, respectively, with $Q_{ij}^{(3)}$ denoting the third time derivative of the mass-quadrupole tensor,

$$\begin{aligned} H_{4\mathrm{PN}}^{\text{near-zone}\,(s)}[\boldsymbol{x}_a,\boldsymbol{p}_a] &= H_{4\mathrm{PN}}^{\mathrm{loc}\,0}[\boldsymbol{x}_a,\boldsymbol{p}_a] + \frac{2}{5}\,\frac{G^2M}{c^8}\left(Q_{ij}^{(3)}\right)^2 \ln\left(\frac{r_{12}}{s_{\mathrm{nz}}}\right), \\ H_{4\mathrm{PN}}^{\mathrm{tail,sym}\,(s)}(t) &= -\frac{1}{5}\,\frac{G^2M}{c^8}\,Q_{ij}^{(3)}(t)\int_{-\infty}^{+\infty} \mathrm{d}v\,\ln\left(\frac{|v|c}{2s_{\mathrm{fz}}}\right) Q_{ij}^{(4)}(t-v)\,. \end{aligned}$$

Matching SF-results for the perturbed Schwarzschild metric from particle in circular motion yields the connection between the analytical regularization scales s_{nz} and s_{fz} in the form of $\ln(s_{\mathrm{fz}}/s_{\mathrm{nz}}) = -\frac{1681}{1536}$, [2], being consistent with

$$\begin{aligned} H_{4\mathrm{PN}}^{\mathrm{tail,nloc}} &= -\frac{GM}{c^3}\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{|\tau|}\mathcal{F}_{\mathrm{N}}^{\mathrm{split}}(t,t+\tau)\,,\ (\text{dim. reg. throughout})\,, \\ \mathcal{F}_{\mathrm{N}}^{\mathrm{split}}(t,t') &= \frac{G}{c^5}\frac{1}{5}Q_{ij}^{(3)}(t)Q_{ij}^{(3)}(t')\,, \\ H_{4\mathrm{PN}}^{\mathrm{tail,loc}} &= -\frac{GM}{c^3}\frac{41}{30}\mathcal{F}_{\mathrm{N}}^{\mathrm{split}}(t,t) \quad [11]\,. \end{aligned}$$

The rational number $\frac{41}{30}$ corresponds to $\frac{5}{6}$ which is a famous number in the Lamb-shift formula, *cf.*, [14].

4.2. The EOB formalism

The EOB-formalism is a most efficient Hamiltonian tool to compile and compare results from various approaches. It is constructed as a PN-series of a binary Hamiltonian in its center-of-mass frame

$$H = m_1c^2 + m_2c^2 + H_N + \sum_{n=1}^{5}\left(\frac{1}{c^2}\right)^n H_{n\mathrm{PN}}\,.$$

The following notions will be used in the following: $H_{\rm NR} = H - Mc^2$, $M = m_1 + m_2$, $\mu = m_1 m_2/M$, $\nu = \mu/M$, $0 \le \nu \le 1/4$, test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$, in center-of-mass frame: $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$, $\boldsymbol{p} \equiv \boldsymbol{p}_1/\mu$, $p_r = (\boldsymbol{n}\cdot\boldsymbol{p})$, $\boldsymbol{q} \equiv (\boldsymbol{x}_1 - \boldsymbol{x}_2)/GM$, $\boldsymbol{n} = \boldsymbol{q}/|\boldsymbol{q}|$, $\hat{t} = t/GM$.

The effective Hamiltonian $H_{\rm eff}$ is defined by

$$\frac{H_{\rm eff}}{\mu c^2} \equiv \frac{H^2 - m_1^2c^4 - m_2^2c^4}{2m_1m_2c^4} = 1 + \frac{H_{\rm NR}}{\mu c^2} + \frac{\nu}{2}\left(\frac{H_{\rm NR}}{\mu c^2}\right)^2 ,$$

resulting in the useful representation of H,

$$H = Mc^2\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}, \qquad \hat{H}_{\rm eff} = \frac{H_{\rm eff}}{\mu c^2}\,.$$

The EOB representation makes use of the generalized Schwarzschild metric

$$\begin{aligned} g^{\mu\nu}_{\rm eff}P_\mu P_\nu + Q_4(P_i) &= -\mu^2c^2\,, \qquad H^{\rm EOB}_{\rm eff} \equiv -P_0 c\,,\\ H^{\rm EOB}_{\rm eff} &= N_{\rm eff}\,c\sqrt{\mu^2c^2 + \gamma^{ij}_{\rm eff}P_iP_j + Q_4(P_i)}\,. \end{aligned}$$

A canonical transformation connects the primary effective Hamiltonian with the effective EOB Hamiltonian

$$H^{\rm EOB}_{\rm eff}(X,P) = H_{\rm eff}(x,p)\,, \qquad H^{\rm EOB} = Mc^2\sqrt{1 + 2\nu\left(\hat{H}^{\rm EOB}_{\rm eff} - 1\right)}\,,$$

$$(X,P) \to (x,p) : \hat{H}^{\rm EOB}_{\rm eff} = \sqrt{A\left(1 + AD\eta^2p_r^2 + \eta^2\left(p^2 - p_r^2\right) + Q\right)}\,, \ (\eta = 1/c)\,.$$

4.2.1. Conservative BBH local-in-time dynamics through 4PN

PN expansions of the functions $A(u)$, $D(u)$, and $Q(u,p_r)$ yield

$$A = 1 + \sum_{k=1}^{6} a_k(\nu)\eta^{2k}u^k\,, \qquad D = 1 + \sum_{k=2}^{5} d_k(\nu)\eta^{2k}u^k\,,$$

$$\begin{aligned} Q = {} & \eta^4p_r^4\left[q_{42}(\nu)\eta^4u^2 + q_{43}(\nu)\eta^6u^3 + q_{44}(\nu)\eta^8u^4\right]\\ & + \eta^6p_r^6\left[q_{62}(\nu)\eta^4u^2 + q_{63}(\nu)\eta^6u^3\right] + \eta^{12}p_r^8q_{82}(\nu)u^2\,, \qquad u = GM/rc^2\,. \end{aligned}$$

At Newtonian level: $a_1 = -2$; at 1PN: $a_2 = 0$; at 2PN: $d_2 = 6\nu, a_3 = 2\nu$; at 3PN: $q_{42} = 8\nu - 6\nu^2, d_3 = 52\nu - 6\nu^2, a_4 = \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu$; at 4PN: $q_{62} = -\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3, q_{43} = 20\nu - 83\nu^2 + 10\nu^3$, $d_4 = \left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + (-260 + \frac{123}{16}\pi^2)\nu^2$, $a_5 = \left(-\frac{4237}{60} + \frac{2275}{512}\pi^2\right)\nu + \left(-\frac{221}{6} + \frac{41}{32}\pi^2\right)\nu^2$.

5. The BBH local-in-time dynamics at 5PN

The TF approach yields [6, 7]

$$
\begin{aligned}
q_{82} &= \frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4 \,, \\
q_{63} &= \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4 \,, \\
q_{44} &= \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{9367}{15} + \frac{31633}{512}\pi^2\right)\nu^2 \\
&\quad + \left(640 - \frac{615}{32}\pi^2\right)\nu^3 \,, \\
d_5 &= \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + d_5^{\nu^2} + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3 \,, \\
a_6 &= \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2} + 4\nu^3 \,,
\end{aligned}
$$

The EFT approach [8, 9], results in, notice right below $q_{44}^{\nu^2\mathrm{ratBMMS}} \neq q_{44}^{\nu^2\mathrm{ratTF}}$,

$$
\begin{aligned}
d_5^{\pi^2\nu^2} &= \frac{306545}{512}\pi^2\nu^2 \,, \qquad a_6^{\pi^2\nu^2} = \frac{25911}{256}\pi^2\nu^2 \,, \\
d_5^{\nu^2} &= \left(-\frac{10442728}{1575} + \frac{306545}{512}\pi^2\right)\nu^2 \,, \\
a_6^{\nu^2} &= \left(-\frac{584881}{525} + \frac{25911}{256}\pi^2\right)\nu^2 \,, \\
q_{44} &= \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{1252924}{1575} + \frac{31633}{512}\pi^2\right)\nu^2 \\
&\quad + \left(640 - \frac{615}{32}\pi^2\right)\nu^3 \,,
\end{aligned}
$$

In [10], TF is confirming BMMS, [8, 9], in part, also comparing with FS and AFS, [11, 12]

$$
\begin{aligned}
H_{5\mathrm{PN}}^{\mathrm{tail,nloc}} &= -\frac{G\left(E/c^2\right)}{c^3}\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{|\tau|}\mathcal{F}_{1\mathrm{PN}}^{\mathrm{split}}(t,t+\tau) \,, \\
\mathcal{F}_{1\mathrm{PN}}^{\mathrm{split}}\left(t,t'\right) &= \frac{G}{c^5}\frac{1}{c^2}\left(\frac{1}{189}O_{ijk}^{(4)}(t)O_{ijk}^{(4)}\left(t'\right) + \frac{16}{45}J_{ij}^{(3)}(t)J_{ij}^{(3)}\left(t'\right)\right) , \\
H_{5\mathrm{PN}}^{\mathrm{tail,loc}} &= -\frac{G\left(E/c^2\right)}{c^3}\Big(R_{\mathrm{oct},e}\mathcal{F}_{1\mathrm{PN}}^{\mathrm{split},MO^2}(t,t) + R_{\mathrm{quad},m}\mathcal{F}_{1\mathrm{PN}}^{\mathrm{split},MJ^2}(t,t)\Big) ,
\end{aligned}
$$

$$
\begin{aligned}
R^{\mathrm{TF}}_{\mathrm{oct},e} &= R^{\mathrm{FS}}_{\mathrm{oct},e} = R^{\mathrm{BMMS}}_{\mathrm{oct},e} = \frac{82}{35}\,,\\
R^{\mathrm{TF}}_{\mathrm{quad},m} &= R^{\mathrm{AFS}}_{\mathrm{quad},m} = R^{\mathrm{BMMS}}_{\mathrm{quad},m} = \frac{147}{60}\,.
\end{aligned}
$$

The current-type quadrupole J_{ij} enters in [10] in the most delicate form

$$
\frac{1}{2} R_{0iab}\epsilon_{abj}J_{ij}\,, \qquad J_{ij} = J_{ji}\,.
$$

Its d-dimensional generalization goes via the avatar $J_{i|ab}$

$$
\begin{aligned}
\epsilon_{abj}J_{ij} &\equiv J_{b|ia}\,, \qquad J_{i|ab} = -J_{b|ai}\,, \qquad J_{i|ab} + J_{a|bi} + J_{b|ia} = 0\,,\\
J^{(3)}_{ij}J^{(3)}_{ij} &\to \frac{1}{2} J^{(3)}_{i|ab}J^{(3)}_{i|ab}\,,\\
J_{i|ab} &= \nu(m_2 - m_1)\left[\left(x^i x^a - \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{d-1}\delta^{ia}\right) v_b \right.\\
&\quad \left. - \left(x^a x^b - \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{d-1}\delta^{ab}\right) v_i - \frac{\boldsymbol{x}\cdot\boldsymbol{v}}{d-1}\left(x^i\delta^{ab} - x^b\delta^{ia}\right)\right]\,.
\end{aligned}
$$

The following relations have been derived in [10], using $R_{\mathrm{oct},e}$ and $R_{\mathrm{quad},m}$

$$
\begin{aligned}
a_6^{\nu^2} &= \frac{25911}{256}\pi^2\nu^2 + \nu^2 R_{a_6}(C_{QQL}, C_{QQQ_1}, C_{QQQ_2})\,, \qquad R_{a_6} = \text{rat. number}\,,\\
d_5^{\nu^2} &= \frac{306545}{512}\pi^2\nu^2 + \nu^2 R_{d_5}(C_{QQL}, C_{QQQ_1}, C_{QQQ_2})\,, \qquad R_{d_5} = \text{rat. number}\,,
\end{aligned}
$$

where R_{a_6} and R_{d_5} are given functions of the constants C defined by actions

$$
\begin{aligned}
S_{QQL} &= C_{QQL}G^2 \int \mathrm{d}t\, Q^{(4)}_{il}Q^{(3)}_{jl}\epsilon_{ijk}L_k\,,\\
S_{QQQ_1} &= C_{QQQ_1}G^2 \int \mathrm{d}t\, Q^{(4)}_{il}Q^{(4)}_{jl}Q_{ij}\,,\\
S_{QQQ_2} &= C_{QQQ_2}G^2 \int \mathrm{d}t\, Q^{(3)}_{il}Q^{(3)}_{jl}Q^{(2)}_{ij}\,,
\end{aligned}
$$

with values all having been calculated in [9] and [11] using in–in (or, closed-time) and in–out formalisms, respectively

$$
\begin{aligned}
C^{\mathrm{FS}}_{QQL} &= -\frac{8}{15} = C^{\mathrm{BMMS}}_{QQL}\,,\\
C^{\mathrm{FS,mem}}_{QQQ_1} &= -\frac{1}{15} = \frac{4}{3}C^{\mathrm{BMMS,mem}}_{QQQ_1}\,,\\
C^{\mathrm{BMMS,cont}}_{QQQ_1} &= \frac{1}{8}\,,\\
C^{\mathrm{FS,mem}}_{QQQ_2} &= -\frac{4}{105} = \frac{4}{3}C^{\mathrm{BMMS,mem}}_{QQQ_2}\,.
\end{aligned}
$$

The abbreviations “mem” and “cont” denote memory and contact terms, respectively.

In [10], the following constraint equation is obtained from the condition on scattering-angles $\chi_4^{\text{cons,EFT}} - \chi_4^{\text{cons,TF}} = 0$ of conservative local-in-time dynamics:

$$0 = \frac{2973}{350} - \frac{69}{2}C_{QQL} + \frac{253}{18}C_{QQQ_1} + \frac{85}{9}C_{QQQ_2}$$

depending on q_{44}. That condition gets fulfilled by neither the values from FS nor those from BMMS. Also in [10], a possibly missing conservative quadratic radiation-reaction (anti-symmetric)^2 term gets mentioned which could lead to the following change of the Hamiltonian [9], $\delta H_{\text{rad}}^{(\text{reac})^2} = a(\eta^5)^2\nu^2\frac{p_r^4}{r^4}$, with $a = -\frac{168}{5}$: inferring $q_{44}^{\text{BMMS}} \to q_{44}^{\text{TF}}$.

6. Beyond 5PN

The TF-approach succeeded with 6PN to some extent [13]

$$\begin{aligned}
H_{5.5\text{PN}}^{\text{tail}^2,\text{nloc}} &= -\frac{107}{210}\frac{G^2\left(E/c^2\right)^2}{c^6}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{\tau}\left[\mathcal{G}^{\text{split}}(t,t+\tau) - \mathcal{G}^{\text{split}}(t,t-\tau)\right] \;[7]\,,\\
\mathcal{G}^{\text{split}}\left(t,t'\right) &= \frac{G}{c^5}\frac{1}{5}Q_{ij}^{(3)}(t)Q_{ij}^{(4)}\left(t'\right)\,,\\
H_{6\text{PN}}^{\text{tail,nloc}} &= -\frac{G(E/c^2)}{c^3}\mathrm{Pf}_{2r_{12}/c}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\tau}{|\tau|}\mathcal{F}_{2\text{PN}}^{\text{split}}(t,t+\tau)\,,\\
\mathcal{F}_{2\text{PN}}^{\text{split}}\left(t,t'\right) &= \frac{G}{c^5}\frac{1}{c^4}\left(\frac{1}{9072}I_{ijkl}^{(5)}(t)I_{ijkl}^{(5)}\left(t'\right) + \frac{1}{84}J_{ijk}^{(4)}(t)J_{ijk}^{(4)}\left(t'\right)\right).
\end{aligned}$$

Not yet confirmed (at 5PN) or calculated (at 6PN) EOB numerical constants are: 3 rational numbers at 5PN (BMMS): $a_6^{\nu^2,\text{rat}}$, $d_5^{\nu^2,\text{rat}}$, $q_{44}^{\nu^2,\text{rat}}$, 8 rational plus nonrational numbers at 6PN (TF): $a_7^{\nu^3}$, $a_7^{\nu^2}$, $d_6^{\nu^2}$, $q_{45}^{\nu^2}$.

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