STIFFNESS, COMPLEXITY, CRACKING AND STABILITY OF RELATIVISTIC COMPACTS STARS*

Ranjan Sharma[†,‡], Satarupa Barman

Department of Physics, Cooch Behar Panchanan Barma University
Cooch Behar 736101, West Bengal, India

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The compactness of a relativistic compact star provides an essential clue to the matter composition of the star. In this paper, we explore many versions of the Tolman VII solution to analyse the maximum compactness of a relativistic star in the context of a given equation of state (EOS). For an EOS specified by the model parameters in the Tolman VII solution, we evaluate the critical bound on compactness above which the stellar composition becomes unstable against radial oscillations. We also outline the possible link between stellar stability, 'complexity', and 'cracking' of an anisotropic stellar configuration.

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1. Introduction

In astronomy and astrophysics, the 'compactness' of a star plays a crucial role in the theoretical modelling of the star. Some stars are so compact that a relativistic treatment is required to describe them. For a self-gravitating compact relativistic star, if the equation of state (EOS) is known, one can obtain the mass–radius (M–R) relationship by integrating the Tolman–Oppenheimer–Volkoff (TOV) equations. Alternatively, the mass and radius of a compact star provide an important clue to its composition. The current era of multi-messenger astronomy facilitates a more accurate estimation of stellar observables such as mass M and radius R, which is expected to constrain the neutron star EOS. In GR, the maximum compactness is provided by the Buchdahl bound, which implies $M/R < 4/9$ [1]. The bound changes when the electromagnetic field is incorporated into the system. In

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† Visiting Associate, The Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India.

‡ Corresponding author: rsharma@associates.iucaa.in

the presence of a charge, the gravitational attraction is counterbalanced by the Coulomb repulsion which prevents the object from collapsing to a point singularity. Sharma *et al.* [2] superimposed the electromagnetic field on uniform density fluid distribution and generated new exact solutions for a static spherically-symmetric charged stellar composition which reduced to a uniform density Schwarzschild solution when the electric field was switched off. Subsequently, they obtained a charged analogue of the Buchdahl bound which has the form

$$u = \frac{M}{R} = \frac{8/9}{\left(1+\sqrt{1-\frac{8\alpha^2}{9}}\right)}\,, \tag{1.1}$$

where $\alpha^2 = Q^2/M^2$ and $Q = q(R)$ is the total charge. The above result also provides an upper bound on $\alpha^2 \leq 9/8$ and $u \leq 8/9 < 1$. For $\alpha^2 = 0$, one regains the original Buchdahl bound $u \leq 4/9$. Earlier, by adopting a different technique, Giuliani and Rothman [4] obtained the same result. Sharma *et al.* [3] also examined the impact of local pressure anisotropy on the maximum compactness of a relativistic star. By demanding that the central pressure must remain finite, their calculation yielded the following bound:

$$\frac{2M}{R} \leq \frac{4(k-2)}{(5k-9)}\,, \tag{1.2}$$

where k is the measure of anisotropy. It is to be stressed here that k is a curvature parameter in the Vaidya–Tikekar stellar model [5], which has a clear geometric interpretation and can be associated with the star's matter composition (EOS). The spacetime metric developed in the Vaidya–Tikekar stellar model is motivated by a geometric property that $t =$ constant hypersurface of the associated spacetime, when embedded in a 4-Euclidean space is not spherical but spheroidal and the parameter k indicates the departure from the sphericity of associated 3-space.

The Tolman VII solution [6] is yet another exact solution of the Einstein field equations that can describe the structure and properties of compact objects like neutron stars. In our work, making use of the Tolman VII solution, we intend to analyse the impact of EOS on the compactness of a star.

2. Different versions of the Tolman VII solution

It is noteworthy that many versions of the Tolman VII solution are available in the literature. We begin by choosing a line element describing the interior of a static, spherically-symmetric star in the standard form

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\,. \tag{2.1}$$

The matter distribution inside the star is assumed to be a perfect fluid described by the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \tag{2.2}$$

where ρ is the energy density of the fluid, p the isotropic pressure, and u^μ is the 4-velocity of the fluid. For the above, the Einstein field equations read as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{-\lambda}-1}{r^2}\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{-\lambda}\nu'}{2r}\right) + \mathrm{e}^{(-\lambda-\nu)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{e}^{\nu}\nu'}{2r}\right) = 0\,, \tag{2.3}$$

$$\mathrm{e}^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 8\pi p\,, \tag{2.4}$$

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho\,, \tag{2.5}$$

where a prime ($'$) denotes derivative with respect to radial coordinate r. The mass function $m(r)$ is defined as

$$\mathrm{e}^{-\lambda} \equiv 1 - \frac{2m}{r}\,. \tag{2.6}$$

In the above and hereafter, we set $G = 1$ and $c = 1$. To integrate the system (2.3)–(2.5), Tolman introduced the following ansatz

$$\mathrm{e}^{-\lambda(r)} = 1 - u\zeta^2\left(5 - 3\zeta^2\right)\,, \tag{2.7}$$

where $\zeta = \frac{r}{R}$. This particular choice of the metric potential is equivalent to choosing an energy-density distribution inside the star as

$$\rho(r) = \rho_{\mathrm{c}}\left(1 - \zeta^2\right)\,, \tag{2.8}$$

where

$$\rho_{\mathrm{c}} = \frac{15M}{8\pi R^3}\,, \tag{2.9}$$

is the central energy density. By integrating the field equations, one obtains the other unknown metric potential as

$$\mathrm{e}^{\nu(r)} = c_1 \cos^2\phi\,, \tag{2.10}$$

with

$$\phi = c_2 - \frac{1}{2}\log\left(\zeta^2 - \frac{5}{6} + \sqrt{\frac{\mathrm{e}^{-\lambda}}{3u}}\right)\,, \tag{2.11}$$

where c_1 and c_2 are integration constants determined from the boundary conditions (continuity of metric functions across the boundary and vanishing of pressure at the boundary). Subsequently, the isotropic pressure is obtained in the form

$$p = \frac{1}{4\pi R^2}\left[\sqrt{3u\,\mathrm{e}^{-\lambda}}\tan\phi - \frac{u}{2}\left(5-3\zeta^2\right)\right] . \tag{2.12}$$

Thus, the Tolman VII solution is a two-parameter $[M,\, \rho_\mathrm{c}]$ family of solutions.

For a more realistic description of neutron stars, Raghoonundun and Hobill [7] considered a more generalized form of the density profile

$$\tilde{\rho}(r) = \rho_\mathrm{c}\left[1-\mu\left(\frac{r}{R}\right)^2\right] , \tag{2.13}$$

where $\mu[0,1]$ is a free parameter representing the 'stiffness' of the EOS of the star [8] which can vary between $0 \leq \mu \leq 1$. In the extreme case of $\mu = 0$, one obtains an incompressible fluid sphere model, and $\mu = 1$ corresponds to the original Tolman VII solution.

With the above energy density profile, the system of equations may be integrated to yield

$$\mathrm{e}^{\tilde{\lambda}} = \frac{1}{1-\left(\frac{8\pi\rho_\mathrm{c}}{3}\right)r^2 + \left(\frac{8\pi\mu\rho_\mathrm{c}}{5R^2}\right)r^4} = \frac{1}{1-br^2+ar^4} , \tag{2.14}$$

$$\mathrm{e}^{\frac{\tilde{\nu}(r)}{2}} = \tilde{c_1}\cos\left(\tilde{\phi}\xi(r)\right) + \tilde{c_2}\sin\left(\tilde{\phi}\xi(r)\right) , \tag{2.15}$$

where $\tilde{\phi} = \sqrt{\frac{a}{4}}$, $\rho_\mathrm{c} = \frac{15M}{4\pi R^3(5-3\mu)}$, and $\xi(r) = \frac{2}{\sqrt{a}}\coth^{-1}\left(\frac{1+\sqrt{1-br^2+ar^4}}{r^2\sqrt{a}}\right)$. One thus obtains a three-parameter $[M,\, \rho_\mathrm{c},\, \mu]$ family of solutions where μ can be fixed to suit a particular EOS.

Jiang and Yagi [9] have proposed an improved version of the Tolman VII solution which is also a three-parameter $[M,\, \rho_v,\, \alpha]$ family of solutions. The motivation for the improved Tolman VII solution was to ensure a better agreement of the energy-density profile with the realistic neutron star EOS. In this approach, the energy density is assumed to be of the form

$$\rho_\mathrm{im}(r) = \rho_\mathrm{c}\left[1-\alpha\zeta^2 + (\alpha-1)\zeta^4\right] , \tag{2.16}$$

where $\alpha[0,2]$ is a free parameter. The original Tolman VII solution has $\alpha = 1$. With this assumption, the metric potentials are obtained as

$$\mathrm{e}^{-\lambda_\mathrm{im}} = 1 - 8\pi R^2\zeta^2\rho_\mathrm{c}\left(\frac{1}{3} - \frac{\alpha}{5}\zeta^2 + \frac{\alpha-1}{7}\zeta^4\right) , \tag{2.17}$$

$$\mathrm{e}^{\nu} = c_1^\mathrm{im}\cos^2\phi_\mathrm{im} , \tag{2.18}$$

with

$$\phi_{\rm im} = c_2^{\rm im} - \frac{1}{2}\log\left(\zeta^2 - \frac{5}{6} + \sqrt{\frac{5e^{-\lambda_{\rm Tol}}}{8\pi R^2 \rho_{\rm c}}}\right). \tag{2.19}$$

The advantages of these generalized versions of the Tolman VII solution are the following:

(i) The parameter μ is a measure of 'stiffness' of the associated EOS;

(ii) The parameter α can be linked to the EOS of the matter composition.

In other words, μ and α can be suitably chosen so as to describe a particular EOS. In our paper, we will examine the effects of these parameters on the stability of the configuration, which can subsequently provide us with an upper bound on compactness.

3. Stability: Chandrasekhar's technique

Chandrasekhar [10], in 1964, proposed a technique to analyse the stability of a star against radial oscillations. Following the technique, Bardeen *et al.* [11] presented a variety of methods to examine the stability of a star numerically. The technique is summarized below.

Let us consider a perturbation in the metric potentials of the line element describing a spherically symmetric star

$$ds^2 = -e^{\nu(r,t)}dt^2 + e^{\lambda(r,t)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.1}$$

as

$$\nu(r,t) = \nu_0(r) + \delta\nu(r,t)\,, \tag{3.2}$$

$$\lambda(r,t) = \lambda_0(r) + \delta\lambda(r,t)\,, \tag{3.3}$$

where ν_0 and λ_0 denote potentials under equilibrium condition, and $\delta\nu(r,t)$ and $\delta\lambda(r,t)$ are small perturbations from its equilibrium condition. Similarly, the perturbed energy density ρ and pressure p are assumed to be of the form

$$\rho(r,t) = \rho_0(r) + \delta\rho(r,t)\,, \tag{3.4}$$

$$p(r,t) = p_0(r) + \delta p(r,t)\,. \tag{3.5}$$

A perturbation in the radial parameter r is assumed to be in the form

$$\delta r = u_n(r)\, e^{\frac{\nu_0(r)}{2}}\, e^{i\omega_n t}/r^2\,, \tag{3.6}$$

where $u_n(r)$ and ω_n are the amplitude and frequency of the $n^{\rm th}$ normal mode of oscillations, respectively.

Utilizing the energy conservation, baryon number conservation principles, and the Einstein field equations, one arrives at the dynamical equation governing the stellar pulsation in its n^{th} normal mode, which has the Sturm–Liouville's form

$$P(r)\frac{\mathrm{d}^2u_n(r)}{\mathrm{d}r^2}+\frac{\mathrm{d}P}{\mathrm{d}r}+\left[Q(r)+{\omega_n}^2W(r)\right]u_n(r)=0\,, \tag{3.7}$$

where the functions $P(r)$, $Q(r)$, and $W(r)$ are expressed in terms of the equilibrium configuration of the star given by

$$P(r) = \gamma p_0\,\mathrm{e}^{\frac{(\lambda_0+3\nu_0)}{2}}r^{-2}\,, \tag{3.8}$$

$$Q(r) = \mathrm{e}^{\frac{(\lambda_0+3\nu_0)}{2}}\left[\frac{{p_0'}^2}{r^2\left(p_0+\rho_0\right)}-\frac{4p_0'}{r^3}-\frac{8\pi p_0}{r^2}\left(\rho_0+p_0\right)\mathrm{e}^{2\lambda_0}\right]\,, \tag{3.9}$$

$$W = \mathrm{e}^{\frac{(3\lambda_0+\nu_0)}{2}}r^{-2}\left(\rho_0+p_0\right)\,. \tag{3.10}$$

For the fundamental mode of oscillation ($n=0$), the pulsation equation takes the form

$$\omega_0^2\int\limits_0^R\exp\left[\frac{1}{2}\left(3\lambda_0+\nu_0\right)\right]\left(p_0+\rho_0\right)\frac{u_0^2}{r^2}\mathrm{d}r=\int\limits_0^R\exp\left(\frac{1}{2}\left(3\nu_0+\lambda_0\right)\right)\left(\frac{p_0+\rho_0}{r^2}\right)$$
$$\left(\left[-\frac{2}{r}+\frac{\mathrm{d}\nu_0}{\mathrm{d}r}-\frac{1}{4}\left(\frac{\mathrm{d}\nu_0}{\mathrm{d}r}\right)^2+8\pi p_0\exp\left(\lambda_0\right)\right]u^2+\frac{\mathrm{d}p_0}{\mathrm{d}\rho_0}\left(\frac{\mathrm{d}u_0}{\mathrm{d}r}\right)^2\right)\mathrm{d}r\,, \tag{3.11}$$

where $\gamma=\frac{p+\rho}{p}\frac{\mathrm{d}p}{\mathrm{d}\rho}$ is the adiabatic index.

A stellar configuration will be stable if ω is real and positive. Since the integration of the left-hand side of equation (3.11) is always positive definite, for stability, the right-hand side of this equation must be positive.

To integrate the right-hand side, we employ the method given by Bardeen *et al.* [11] and assume a trial solution for u_0 as

$$u_0=r\,\mathrm{e}^{\frac{\nu_0(r)}{2}} \tag{3.12}$$

with the following boundary conditions:

(i) $u_0\approx r^3$ as $r\to 0$,

(ii) the Lagrangian change in pressure (Δp) at the surface ($r=R$) must vanish which implies $\frac{\mathrm{d}u_0}{\mathrm{d}r}\to 0$ as $r\to R$.

In the following section, we utilize the above technique to analyze the stability of a particular stellar configuration.

4. Numerical analysis

We evaluate the right-hand side of equation (3.11) by *(i)* increasing the mass for a fixed radius star (we take $R = 10$ km) and *(ii)* decreasing the radius for a fixed mass star (we take $M = 1.4\, M_{\odot}$). The results are given below in a tabular form.

4.1. Stability range in the case of standard Tolman VII solution

In Tables 1 and 2, it is interesting to note that the configuration in this model remains stable for maximum compactness ~ 0.38 which is below the Buchdal bound ($\frac{M}{R} < \frac{4}{9}$).

Table 1. Compactness bound below which a stellar configuration remains stable (standard Tolman VII solution with $\mu = 1$). The radius is kept fixed at $R = 10$ km.

Mass (M) [$M_{\odot}$]	Compactness (M/R)	Integral
1.4	0.2065	$+ve$
1.6	0.236	$+ve$
1.8	0.2655	$+ve$
2	0.295	$+ve$
2.2	0.3245	$+ve$
2.4	0.354	$+ve$
2.5	0.3695	$+ve$
2.6	0.3835	$-ve$

Table 2. Compactness bound below which a stellar configuration remains stable (standard Tolman VII solution with $\mu = 1$). The mass is kept fixed at $M = 1.4\, M_{\odot}$.

Radius (R) [km]	Compactness (M/R)	Integral
10	0.2065	$+ve$
9	0.2294	$+ve$
8	0.2581	$+ve$
7	0.295	$+ve$
6	0.3441	$+ve$
5.3	0.3896	$-ve$

4.2. Stability range in the case of generalized Tolman VII solution

The generalized Tolman VII solution admits a wide range of values of the stiffness parameter μ and hence the solution permits us to analyse the impact of 'stiffness' on the stability of a given configuration. The numerical results are summarized in Tables 3–5. In Table 6, we compile the results to show the impact of the 'stiffness' parameter on the upper bound of compactness from the stability point of view.

Table 3. Compactness bound below which a stellar configuration remains stable (generalized Tolman VII solution with $\mu = 0.1$). The radius is kept fixed at $R =$ 10 km.

Mass (M) $[M_\odot]$	Compactness (M/R)	Integral
1.4	0.2065	$+ve$
1.8	0.2655	$+ve$
2.2	0.3245	$+ve$
2.6	0.3835	$+ve$
2.8	0.413	$+ve$
3	0.4425	$+ve$
3.1	0.45725	$-ve$

Table 4. Compactness bound below which a stellar configuration remains stable (generalized Tolman VII solution with $\mu = 0.6$). The radius is kept fixed at $R =$ 10 km.

Mass (M) $[M_\odot]$	Compactness (M/R)	Integral
1.4	0.2065	$+ve$
1.8	0.2655	$+ve$
2.2	0.3245	$+ve$
2.6	0.3835	$+ve$
2.8	0.413	$+ve$
2.9	0.42775	$-ve$

4.3. Stability analysis using the improved Tolman VII solution

The improved Tolman VII solution admits a wide range of values of the EOS parameter α. A similar analysis yields a relationship between α and u which is compiled in Table 7.

Table 5. Compactness bound below which a stellar configuration remains stable (generalized Tolman VII solution with $\mu = 0.9$). The mass is kept fixed at $M = 1.4\, M_{\odot}$.

Radius (R) [km]	Compactness (M/R)	Integral
10	0.2065	$+ve$
9	0.229	$+ve$
8	0.258	$+ve$
7	0.295	$+ve$
6	0.344	$+ve$
5	0.413	$-ve$

Table 6. Stiffness parameter (μ) and the critical bound on compactness $(M/R)_{\mathrm{max}}$.

μ	$(M/R)_{\mathrm{max}}$
0.1	0.44075
0.6	0.417
0.9	0.3938
1	0.37245

Table 7. Relationship between the EOS parameter α and critical bound on compactness $(M/R)_{\mathrm{max}}$.

α	u_{critical}
0.8	0.439362
1	0.382407
1.5	0.338525

5. Discussions

Our results show a correlation between stiffness *vis-a-vis* EOS of the matter composition and the stability of a stellar configuration. With the departure from homogeneous or constant density star ($\mu = 0$), the critical upper bound on compactness decreases.

It is obvious for a self-gravitating star that the simplest possible configuration is a homogeneous fluid distribution with isotropic pressure. Such a configuration is assumed to have zero 'complexity'. Similarly, an anisotropic stellar configuration (having unequal principal stresses) with inhomogeneous energy-density distribution can also have zero 'complexity' provided the terms denoting density inhomogeneity and anisotropic pressure cancel out each other. Consequently, following the concept of 'cracking' put forward by Herrera [12], Abreu *et al.* [13] proposed a criterion which could be used to identify a potentially stable region within a stellar composition. It is observed that the region having $-1 \leq v_{\mathrm{st}}^2 - v_{\mathrm{sr}}^2 \leq 0$ would be a potentially stable region, whereas the region for which $0 < v_{\mathrm{st}}^2 - v_{\mathrm{sr}}^2 \leq 1$ is expected to be potentially unstable, where $v_{\mathrm{st}}^2 = \frac{\mathrm{d}p_{\mathrm{t}}}{\mathrm{d}\rho}$, $v_{\mathrm{sr}}^2 = \frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}\rho}$. Ratanpal [14] has recently shown that a spherically symmetric anisotropic matter distribution would be potentially stable provided the gradient of anisotropy p_t–p_r remains an increasing function of the radial variable r. All these observations point towards an intricate relationship between EOS, 'complexity' and stability of a stellar configuration. We have made some progress in this direction which will be reported elsewhere.

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