MAXIMUM MASS AND STABILITY OF DIFFERENTIALLY ROTATING NEUTRONS STARS*

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Received 12 February 2023, accepted 3 April 2023, published online 13 June 2023

We present our study on stability of differentially rotating, axisymmetric neutron stars described by a polytropic equation of state with $\mathit{\Gamma} = 2$. We focus on quasi-toroidal solutions with a degree of differential rotation $\widetilde{A} = 1$. Our results show that for a wide range of parameters, hypermassive, quasi-toroidal neutron stars are dynamically stable against quasi-radial perturbations, which may have implications for newly born neutron stars and binary neutron stars mergers.

DOI:10.5506/APhysPolBSupp.16.6-A8

1. Introduction

Differential rotation seems to appear naturally in many dynamical scenarios involving neutron stars (NS), including the collapse of stellar cores (see *e.g.* [1]) and binary neutron star (BNS) mergers (see *e.g.* [2]). Its stabilizing effect may allow for configurations with masses significantly higher than the mass limit for rigidly rotating neutron stars. Its study is relevant for understanding of the black-hole formation in those astrophysical scenarios with consequences in observations of core-collapse supernovae (CCSN) and BNS mergers, especially with current gravitational wave ground-based observatories (LIGO, Virgo, and KAGRA [3–5]) and future ones (the Einstein Telescope and Cosmic Explorer).

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* Presented at the 8$^{\rm th}$ Conference of the Polish Society on Relativity, Warsaw, Poland, 19–23 September, 2022.

1.1. Equilibrium models of differentially rotating NS

The solution space of differentially rotating neutron stars in equilibrium was already extensively studied by different authors. It was shown in [6] that differentially rotating NS with masses significantly larger than non-rotating or rigidly rotating NS can exist and be stable against radial collapse and bar formation. Those with masses larger than the limit for rigidly rotating objects are called hypermassive NS.

However, studying the whole solution space has proven to be numerically challenging. The existence of different types of solutions of differentially rotating neutron stars was for the first time found by [7] using relativistic highly accurate and stable multi-domain spectral numerical code FlatStar. Most importantly, for a given degree of differential rotation, the solution is not uniquely determined by the maximal density and angular momentum of the NS (or any other suitable pair of parameters), as is the case for rigid rotation. Instead, different types of solutions may coexist for the same parameters. The maximum mass for different degrees of differential rotation and different solution types was presented in [8] and [9] for polytropes, showing that the most massive configurations are obtained for a modest degree of differential rotation. Similar results were obtained for strange quark stars [10] and NS with several realistic equations of state [11].

While many studies in the past were using a rotation law of Komatsu, Eriguchi, and Hachisu [12], which is mainly consistent with CCSN remnants [1], the rotation law observed in simulations of BNS merger remnants departs significantly from that one. Rotation laws better suited for BNS mergers have been proposed in [13], and their impact on the solution space of equilibrium models studied in [14].

1.2. Stability properties of hypermassive NS

For non-rotating NS, the limit for both secular and dynamical stability occurs at the point of maximal mass (M_{TOV}). This criterion can be, to some point, extended to rigidly rotating NS. The so-called turning point criterion was presented by Friedman, Ipser, and Sorkin [15], and proven to be a sufficient criterion for instability. It states that the point of maximal gravitational mass M on a sequence of configurations of fixed angular momentum J (J-constant turning points), or, alternatively, the point of minimal gravitational mass on a sequence of fixed rest mass M_0 (M_0-constant turning points) marks the onset of instability. This criterion, however, does not give the exact threshold to collapse. The neutral-stability point where F-mode frequency vanishes differs from the turning-point line [16]. Numerical simulations confirm that the neutral-stability line marks the threshold to prompt collapse.

For rigidly rotating NS, the J-constant turning points coincide with the M_0-constant turning points, but it is no longer the case for differential rotation of a given degree. While other authors usually refer to the former, in this paper, we use the latter as we find it to be a closer estimate of the stability threshold.

On secular timescales, differential rotation transforms into rigid rotation due to the effects of viscosity and magnetic breaking [17–19]. By definition, hypermassive NS have masses that cannot be supported by rigid rotation only. This eventually may lead to a delayed collapse and delayed emission of gravitational waves. There is no clear criterion of dynamical stability for hypermassive NS to tell if the collapse will be prompt or delayed.

Various authors have studied the stability properties of differentially rotating NS by means of numerical simulations. An example of hypermassive NS dynamically stable against both radial instabilities and bar formation was presented by [6]. In [20], the authors explore the limit of stability to quasi-radial oscillations for differentially rotating NS, excluding quasi-toroidal configurations. The threshold to collapse proves to be close to the (J-constant) turning-point line, which is still a valid sufficient criterion of dynamical instability. A caveat for the large masses supported by many of these works is that they may be subject to non-axisymmetric corrotational instabilities (usually known as low-$T/|W|$ instabilities, see *e.g.* [21]) that are able to transport efficiently angular momentum and erase differential rotation. Although there is no clear criterion for the onset of this instability, all studied cases in the literature of NS with quasi-toroidal shape (*e.g.* [22]) have shown the dynamical growth of these instabilities.

2. Equilibrium models

We consider the axisymmetric, stationary configuration of rotating fluid in cylindrical coordinates (t, ρ, z, ϕ). The configurations we study are highly flattened and cylindrical coordinates are more practical than spherical ones. The line element associated with such a configuration may be written in the form of

$$\mathrm{d}s^2 = -\mathrm{e}^{2\nu}\mathrm{d}t^2 + \mathrm{e}^{2\mu}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + W^2\mathrm{e}^{-2\nu}(\mathrm{d}\phi - \omega\mathrm{d}t)^2\,, \tag{1}$$

with four metric potentials μ, W, ν, ω, being functions of ρ and z only, due to symmetry.

In general, the properties of matter are defined by the equation of state (EOS). Here, we use a polytropic EOS with $\Gamma = 2$ (or $N = 1$ in alternate notation) which yields a relation between total mass–energy density ϵ and pressure p

$$\epsilon(p) = p + \sqrt{\frac{p}{K}}\,, \tag{2}$$

where K is the polytropic constant.

We often use a dimensionless value of relativistic enthalpy as a main thermodynamical parameter, which in the case of polytrope with $\Gamma = 2$ can be expressed as

$$H = \log\left(1 + 2K\epsilon_B\right) , \tag{3}$$

where ϵ_B is the rest-mass density.

To describe the differential rotation profile, one needs to specify the rotation law. For the fluid four-velocity u^α and angular velocity Ω, it can be defined as a function $u^t u_\phi = F(\Omega)$. Here, we use the KEH rotation law [15]

$$F(\Omega) = A^2(\Omega_c - \Omega) , \tag{4}$$

with Ω_c being the angular velocity on the rotation axis and A a constant describing the steepness of the profile. This produces a rotation profile consistent with remnants of CCSN. As a dimensionless measure of the degree of differential rotation, we use the value of $\widetilde{A} = \frac{r_{\mathrm{e}}}{A}$ with r_{e} being the star radius on the equatorial plane.

To construct the initial data, we solve the relativistic field equations for four metric potentials on the ρ, z plane using the highly accurate code FlatStar. It uses an efficient multi-domain spectral method to construct equilibrium models of rotating compact objects. For more technical details, see [7] and the appendix of [8].

In this paper, we focus on the degree of differential rotation $\widetilde{A} = 1$. According to the classification of [7], all configurations studied here are of type C. Our main interest lies in quasi-toroidal configurations, which produce the largest masses and are not extensively studied by other authors. Figure 1 shows the relativistic enthalpy profile of one of such configurations.

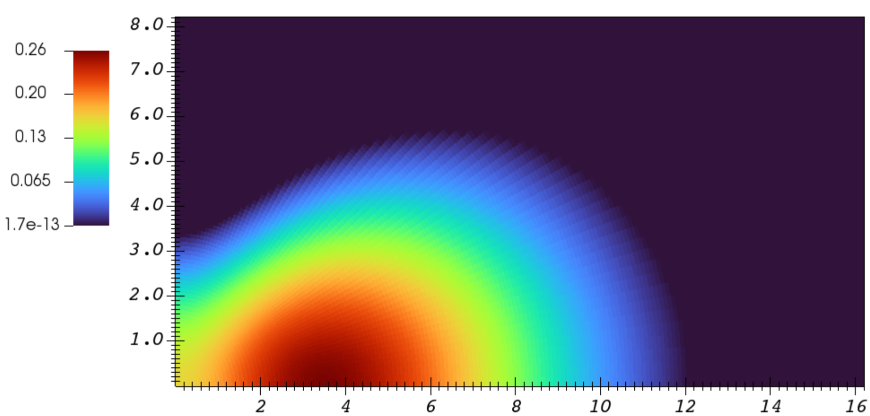

Fig. 1. Example of a quasi-toroidal initial configuration ($H_{\mathrm{max}} = 0.26$, $M_0 = 0.33$). Color coded, the relativistic enthalpy H in a meridional cross section. The maximal value H_{max} is found far off the rotation axis ($y = 0$).

The maximal value, $H_{\rm max}$, is not in the center of the star. We select 5 sequences with a constant rest mass of configurations close to the stability limit estimated by the turning point criterion.

3. Stability against quasi-radial instabilities

To test the stability of the selected configurations, we perform relativistic axisymmetric hydrodynamical simulations using the CoCoNuT code [23], which uses the conformally flat approximation (CFC). In the 3 + 1 split, the line element reads

$$\mathrm{d}s^2 = -\alpha^2 \mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i \mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j \mathrm{d}t\right) , \tag{5}$$

where $\gamma_{ij} = \Phi^4 \delta_{ij}$ in the CFC approximation, with the conformal factor $\Phi^6 = \mathrm{e}^{2\mu - u}$ (using variables from equation (1)). The accuracy of CFC was tested, for example, in [24], showing that the astrophysical properties (such as mass) are reconstructed with only a small discrepancy less than 5%.

To induce the collapse in unstable configurations, we introduce a radial perturbation in the form of an additional velocity component. The amplitude of this perturbation is carefully chosen to make the mean value of the

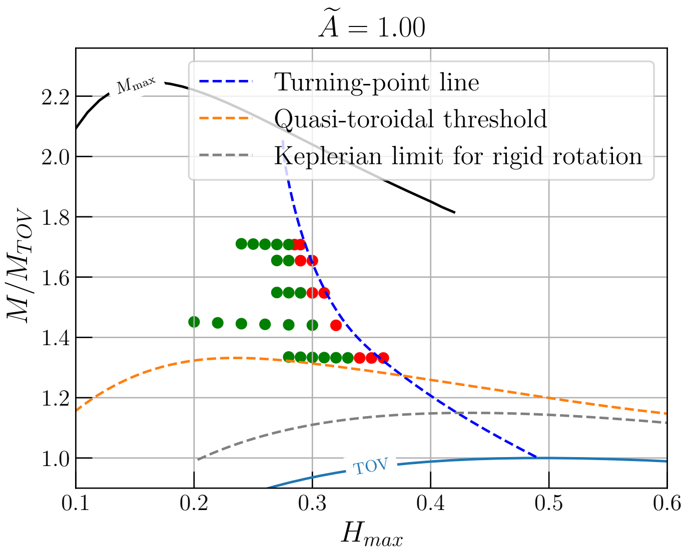

Fig. 2. Simulated configurations divided into stable (green marks) and unstable (red marks) to quasi-radial perturbations. The blue dashed line shows the line of (M_0-constant) turning points, being the first estimate of stability. The orange dashed line marks the boundary between spheroidal and quasi-toroidal configurations. The limit of mass for this degree of differential rotation, the limit for rigid rotation, and the sequence of non-rotating NS are presented for reference.

maximal density match the initial value in stable solutions (for unstable solutions, we extrapolate the amplitude from the stable region). We test our results by comparing them with the results of [20] and [16], finding them to be in agreement.

For all our models, we inspect the evolution of maximal density in time. For stable configurations, we see the value of density oscillating around the initial value. Unstable configurations show an exponential growth of the maximal density in the first few ms, marking the prompt collapse to a BH. Figure 2 shows stable and unstable configurations on the H_{max}–M plane.

4. Summary

We have selected a sample of quasi-toroidal configurations of neutron stars with the polytropic equation of state. We used a polytrope with $\Gamma = 2$ and KEH rotation law. By performing a numerical relativistic hydrodynamical evolution, we tested the stability of the selected equilibria against axisymmetrical (quasi-radial) perturbations. We show that differential rotation allows the existence of dynamically stable models with masses almost twice as massive as M_{TOV}. These stable configurations, if formed during CCSN or BNS mergers, may undergo a significantly delayed collapse into a black hole. Further study is needed to inspect the stability properties against non-axisymmetrical perturbations on dynamical timescales.

This work was partially supported by the National Science Centre (NCN), Poland grants No. 2017/26/M/ST9/00978 and 2022/45/N/ST9/04115, by POMOST/2012-6/11 Program of Foundation for Polish Science co-financed by the European Union within the European Regional Development Fund, by the Spanish Agencia Estatal de Investigación (grants No. PGC2018-095984-B-I00 and PID2021-125485NB-C21) funded by MCIN/AEI/10.13039/501100011033 and ERDF A way of making Europe, by the Generalitat Valenciana (PROMETEO/2019/071), and by COST Actions CA16104 and CA16214.

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