ENERGY-MOMENTUM TENSOR OF A HYDROGEN ATOM: STABILITY, D-TERM, AND THE LAMB SHIFT*

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We clarify two issues related to the so-called D-term in matrix elements of the energy-momentum tensor. First, we show that in a stable system, the D-term can have either sign, contrary to claims that it must be negative. Second, we demonstrate a logarithmic enhancement of the α correction to the D-term in any state of the hydrogen atom. We contrast this enhancement with the Lamb shift where it is present only in S-states.

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1. Introduction

Gravitational interaction of a particle is governed by its energy-momentum tensor (EMT), $T^{\mu\nu}$. Although gravitational interactions are too weak for present experiments to probe them, distributions of mechanical properties encoded in the EMT, such as the energy density, shear, and pressure, can be revealed in scattering processes. For example, a very recent study [1] determined the gluonic contribution to gravitational form factors (GFFs) of the proton.

The GFFs were introduced in [2] and were subsequently studied in [3]. Radiative corrections to graviton–matter interaction were studied in [4]. Recently, GFFs have attracted attention because indirect measurements are possible through generalized parton distributions (GPD) [5] in processes like deeply virtual Compton scattering (DVCS) [6]. Experiments like the measurements of the gluonic and quark contributions to the GFFs [1, 7] are ongoing in JLab and are planned in the future Electron–Ion Collider in Brookhaven.

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For a spin-0 particle without internal structure, the matrix element of $T^{\mu\nu}$ between states with momenta $p_1 = p + \frac{q}{2}$, $p_2 = p - \frac{q}{2}$ is (we use $\hbar = c = 1$)

$$\langle p_2 \left| T^{\mu\nu}(x) \right| p_1 \rangle = \left[2p^\mu p^\nu - \tfrac{1}{2}\left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) \right] \mathrm{e}^{i(p_2-p_1)x} \,. \tag{1}$$

If the internal degrees of freedom of the system become relevant at accessible energies, the two tensor structures are modified by q^2-dependent form factors (see, for example, Ref. [8])

$$\langle p_2 \left| T^{\mu\nu}(x) \right| p_1 \rangle = \left[A\left(q^2\right) p^\mu p^\nu + \tfrac{1}{2} D\left(q^2\right) \left(q^\mu q^\nu - q^2 g^{\mu\nu} \right) \right] \mathrm{e}^{i(p_2-p_1)x} \,. \tag{2}$$

In the present paper, we are particularly interested in the D-term [8], which we discuss with the example of a spin-0 system. Examples include a pion (discussed in this conference [9]) or a hydrogen atom.

Comparing Eqs. (1) and (2), we see that $D \equiv D\left(q^2 = 0\right) = -1$. It has been conjectured that in any stable system, the D must be negative [10]. This is indeed the case for Q-balls and Q-clouds studied in [11, 12], for a liquid drop model considered in [8], for the bag model of the nucleon in [13], and for the chiral quark soliton model [14]. The D-term for spin-0 point-like and composite particles is discussed in [15] and the dynamic origin of the D-term for a spin-1/2 fermion in [15]. The D-term for pion in relativistic theory is calculated in [16].

However, Ref. [17] provided a counterargument that the D-term can also be positive without endangering mechanical stability. In Section 2, we review Max von Laue's stability criterion and discuss its connection with the sign of the D-term. In Section 3 we discuss this sign with an example taken from the classical mechanics.

In Sections 4 and 5, we focus on the hydrogen atom since it can be studied analytically. In particular, it has been found that radiative effects in $D\left(q^2\right)$ are enhanced logarithmically [17, 18]. In Section 4, we review the logarithmic Lamb shift correction. Finally, in Section 5, we contrast it with the logarithmic correction to the D-term.

2. von Laue's stability criterion

In a static system, the conservation law $\partial^\mu T_{\mu\nu} = 0$ has only spatial derivatives

$$\nabla^i T_{i\nu} = 0 \,. \tag{3}$$

Momentum density T_{i0} vanishes in a static system, so we consider only $\nu = j$. von Laue proposed [19, 20] to integrate Eq. (3) over a surface consisting of a cross section σ of the static system and closing far away, where T_{ij} is assumed to vanish. We repeat here von Laue's reasoning that led him to formulate a criterion of stability of a system.

We work in the system's rest frame. One finds that the following integral over the cross section vanishes

$$\int\limits_{\sigma} T^{ij} n_j \mathrm{d}\sigma = 0\,, \tag{4}$$

where n_j are the components of a vector normal to the cross section. The closing part of the surface does not contribute to the integral. Equation (4) becomes a system of three equations for $i = x, y, z$

$$\int T^{xi} \mathrm{d}y\,\mathrm{d}z = 0\,. \tag{5}$$

We integrate this over all x

$$\int T^{xi} \mathrm{d}^3 r = 0\,. \tag{6}$$

We conclude that each component T^{xi}, and in general each T^{ij}, $i, j = x, y, z$, integrated over the whole volume of the system gives zero. This is von Laue's stability condition.

As a non-trivial application, consider a drop of water in vacuum in conditions of weightlessness [8]. Due to surface tension σ, there is pressure inside the drop, $p(r) = 2\sigma/R$ for $r < R$, where R is the radius of the drop (this is Laplace's formula [21]). Pressure p is a diagonal element T^{ii} (in an isotropic system all three such elements are the same). The stability condition (6) tells us that

$$\int p(r) \mathrm{d}^3 r = 4\pi \int\limits_0^R p(r) r^2 \mathrm{d}r = 0\,. \tag{7}$$

This vanishing of the integral is of course only possible if $p(r)$ is negative somewhere. There is a thin surface layer where surface tension makes T^{ii} negative precisely to such an extent that it cancels the positive contribution of the bulk. This tension holds the drop together.

On the other hand, the D-term can be expressed as [8] (we assume spherical symmetry)

$$D = 4\pi m \int p\,(r)\, r^4 \mathrm{d}r\,, \tag{8}$$

which has two additional powers of r in comparison with Eq. (7). This gives larger weight to $p\,(r)$ at larger values of r. If, as in the case of a liquid drop, the negative contribution comes from the outer boundary, a negative D results. On the other hand, other binding mechanisms may have negative p at short distances, leading to a positive D. This is illustrated in the next section.

3. The energy-momentum tensor for a classical system

A negative D-term is unusual in classical mechanics. To demonstrate this, we consider the system composed of a static charge $e = \sqrt{4\pi\alpha} > 0$ at $r = 0$ (a nucleus) and a negatively charged point-like particle in circular motion around it with radius R and velocity $mv^2 = \frac{\alpha}{R}$ (we use such units that $\hbar = c = \epsilon_0 = 1$). The EMT of the system reads

$$T^{ij}(\vec{r}) = mv^i v^j \delta^3\left(\vec{r} - \vec{x}(t)\right) - E^i E^j + \frac{\delta^{ij}}{2}\vec{E}^{\,2}\,, \tag{9}$$

where the electric field is approximated by the static Coulomb fields of the nucleus and of the orbiting particle

$$\vec{E} \approx \frac{e}{4\pi}\frac{\vec{r}}{r^3} - \frac{e}{4\pi}\frac{\vec{r} - \vec{x}(t)}{|\vec{r} - \vec{x}(t)|^3} \equiv \vec{E}_p + \vec{E}_e\,. \tag{10}$$

We first show that $\int \mathrm{d}^3\vec{r}\, T^{ii}(\vec{r}) = 2T + V = 0$. Indeed,

$$\int \mathrm{d}^3\vec{r}\, T^{ii}(\vec{r}) = mv^2 + \frac{1}{2}\int \mathrm{d}^3\vec{r}\vec{E}^{\,2}\,, \tag{11}$$

where

$$\frac{1}{2}\int \mathrm{d}^3\vec{r}\, \vec{E}^{\,2} = \frac{1}{2}\int \mathrm{d}^3\vec{r}\left(\vec{E}_e^{\,2} + \vec{E}_p^{2}\right) + \int \mathrm{d}^3\vec{r}\, \vec{E}_e \cdot \vec{E}_p\,. \tag{12}$$

The self-energy contributions vanish in dimensional regularization (DR), while the last integral simply gives the Coulomb potential

$$\int \mathrm{d}^3\vec{r}\, \vec{E}_e \cdot \vec{E}_p = -\frac{\alpha}{|\vec{x}(t)|} = -\frac{\alpha}{R}\,. \tag{13}$$

Therefore, one simply has

$$\int \mathrm{d}^3\vec{r}\, T^{ii}(\vec{r}) = mv^2 - \frac{\alpha}{R} = 2T + V = 0 \tag{14}$$

equivalent to the virial theorem. We now move to the $\int \mathrm{d}^3\vec{r}\, r^2 T^{ii}$ integral. We first consider

$$I = \int \mathrm{d}^3\vec{r} r^2\left(\frac{1}{r^4} + \frac{1}{|\vec{r} - \vec{R}|^4} - \frac{2r^2 - 2\vec{r}\cdot\vec{R}}{r^3|\vec{r} - \vec{R}|^3}\right)\,, \tag{15}$$

using $r^2 = R^2 + |\vec{r}-\vec{R}|^2 + 2\vec{R}\cdot\left(\vec{r}-\vec{R}\right)$ in the second term and $r^2 - \vec{r}\cdot\vec{R} = |\vec{r}-\vec{R}|^2 + \vec{R}\cdot(\vec{r}-\vec{R})$ in the third term, one has

$$\begin{aligned} I &= \int \mathrm{d}^3\vec{r}\left(\frac{1}{r^2} + \frac{1}{|\vec{r}-\vec{R}|^2} + \frac{R^2}{|\vec{r}-\vec{R}|^4} + \frac{2\vec{R}\cdot\left(\vec{r}-\vec{R}\right)}{|\vec{r}-\vec{R}|^4}\right. \\ &\quad \left. -\frac{2}{|\vec{r}||\vec{r}-\vec{R}|} + \frac{2\vec{R}\cdot\left(\vec{r}-\vec{R}\right)}{r|\vec{r}-\vec{R}|^3}\right) . \end{aligned} \tag{16}$$

The first four terms all vanish in DR, leaving only

$$I = I_1 + I_2 \equiv \int \mathrm{d}^3\vec{r}\left(-\frac{2}{|\vec{r}||\vec{r}-\vec{R}|} + \frac{2\vec{R}\cdot\left(\vec{r}-\vec{R}\right)}{r|\vec{r}-\vec{R}|^3}\right) . \tag{17}$$

In DR the above can be further calculated as

$$\begin{aligned} I_1(D) &= -\frac{2}{\pi}\int \frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2}{\sqrt{\alpha_1\alpha_2}} \int \mathrm{d}^D\vec{r}\,\mathrm{e}^{-(\alpha_1+\alpha_2)r^2 - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}R^2} \\ &\to 4\pi R|_{D=3} , \end{aligned} \tag{18}$$

$$\begin{aligned} I_2(D) &= -2R^2\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{\pi}}\int \frac{\mathrm{d}\alpha_1\mathrm{d}\alpha_2\sqrt{\alpha_1\alpha_2}}{\alpha_1+\alpha_2}\int \mathrm{d}^D\vec{r}\,\mathrm{e}^{-(\alpha_1+\alpha_2)r^2 - \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}R^2} \\ &\to -4\pi R . \end{aligned} \tag{19}$$

Therefore, one simply has

$$\int \mathrm{d}^3\vec{r}\, r^2\, T^{ii}(\vec{r}) = mv^2R^2 + \frac{e^2}{32\pi^2}\left(I_1 + I_2\right) = \alpha R , \tag{20}$$

or $\tau = \frac{1}{12}\int \mathrm{d}^3\vec{r}\, r^2\, T^{ii}(\vec{r}) = \frac{\alpha R}{12} > 0$. If one uses $R \sim \langle r\rangle = \frac{3}{2\alpha m}$, then $\tau \sim \frac{1}{8m}$, comparing to the leading order (LO) result $\tau_H = \frac{1}{4m}$ (see Eq. (61) in Ref. [18]).

4. Logarithmic enhancement of α corrections to the Lamb shift

In this section, we summarize Welton's heuristic explanation of the leading part of the Lamb shift [22].

Welton argued that, although the expectation value of the electromagnetic field strength vanishes in vacuum, there are non-zero fluctuations such that $E_k^2 = \langle 0\left|E_{\boldsymbol{k}}^2\right|0\rangle \neq 0$ for all modes, where $\boldsymbol{k}$ denotes a wave vector.

Consider the system to be enclosed in a large cube of volume V, with periodic boundary conditions. Since the vacuum energy of one mode is $\omega_k/2$, where $\omega_k = |\boldsymbol{k}|$, and on the other hand, the energy density is related to the squares of the electric and the magnetic fields, we obtain

$$E_k^2 = \frac{\omega_k}{2V} \,. \tag{21}$$

$E_{\boldsymbol{k}}$ is the amplitude of a plane wave, $E_{\boldsymbol{k}} \exp i\boldsymbol{k}\cdot\boldsymbol{r}$. The mode density of such running plane waves is

$$\frac{V\mathrm{d}^3k}{(2\pi)^3} \,. \tag{22}$$

Due to the fluctuating electric field, the electron's position is modified and this displacement in the Coulomb potential $U_{\mathrm{C}}(r) = -\alpha/r$ gives rise to an extra effective potential δU

$$\langle U_{\mathrm{C}}(\boldsymbol{r}+\boldsymbol{q})\rangle = U_{\mathrm{C}}(r) + \underbrace{\langle \boldsymbol{q}\rangle}_{0} \cdot \boldsymbol{\nabla} U_{\mathrm{C}} + \frac{1}{2}\underbrace{\left\langle q^i q^j \right\rangle}_{\frac{\delta_{ij}}{3}\langle q^2\rangle} \nabla^i\nabla^j U_{\mathrm{C}} + \ldots, \tag{23}$$

$$\delta U = \langle U_{\mathrm{C}}(\boldsymbol{r}+\boldsymbol{q})\rangle - U_{\mathrm{C}}(r) \simeq \frac{1}{6}\left\langle q^2\right\rangle \nabla^2 U_{\mathrm{C}} = \frac{1}{6}\left\langle q^2\right\rangle \alpha 4\pi\delta^3(\boldsymbol{r}) \,. \tag{24}$$

In order to find the mean disturbance $\left\langle q^2\right\rangle$, Welton considered the equation of motion, assuming a free electron

$$m\ddot{q} = -eE \,, \tag{25}$$

where E is the electric field. In the Fourier space, for each mode $\boldsymbol{k}$,

$$-m\omega^2 q_k = -eE_k \to q_k = \frac{e}{m\omega^2} E_k \,. \tag{26}$$

The total disturbance, summed over all modes, including a factor 2 for two polarizations,

$$\left\langle q^2\right\rangle = 2\int \left(\frac{e}{m\omega^2}\right)^2 \frac{V\mathrm{d}^3k}{(2\pi)^3} E_{\boldsymbol{k}}^2 = \frac{2\alpha}{\pi m^2}\int \frac{\mathrm{d}k}{k} \,. \tag{27}$$

For small k, this integral is cut off by energies of the order of excitations of the atom, where the electron cannot be treated as free: $k_{\mathrm{min}} \sim \alpha^2 m$. For large k, it is cut off at the inverse Compton wavelength of the electron: when the electron absorbs a large momentum, it becomes relativistic and its increased inertia decreases its displacement. The upper limit, $k_{\mathrm{max}} \sim m$, does not contain α. Thus, the logarithmic dependence on α is determined by the lower limit

$$\left\langle q^2 \right\rangle = \frac{2\alpha}{\pi m^2} \ln \frac{1}{\alpha^2} + \text{non-logarithmic terms}\,. \tag{28}$$

Substituting this into (24), we find

$$\delta U = \frac{8}{3} \frac{\alpha^2}{m^2} \ln \frac{1}{\alpha} \cdot \delta^3 \left(\boldsymbol{r} \right) \,. \tag{29}$$

For example, in the $2S$ state, $\left\langle \delta^3 \left(\boldsymbol{r} \right) \right\rangle_{2S} = \psi_{2S}^2 \left(0 \right) = 1/8\pi a_{\mathrm{B}}^3$ where a_{B} is the Bohr radius. That state's energy changes by

$$\langle \delta U \rangle_{2S} = \frac{m}{3\pi} \alpha^5 \ln \frac{1}{\alpha} \,. \tag{30}$$

This energy corresponds to the frequency of about 1000 MHz, which is the shift observed by Lamb and Retherford [23]. We stress that the perturbing potential δU in Eq. (24) is proportional to the Laplacian of the Coulomb potential, which vanishes except where the electric charge is present, that is in the nucleus. For this reason, this mechanism, giving rise to the logarithmic enhancement, applies only to S-states (vanishing angular momentum).

5. Logarithmic correction to D-term and effective theory

Contrary to the logarithmic correction to the Lamb shift, which depends on $\nabla^2 V$, the NLO logarithm for the D-term is almost universal. Indeed, the logarithmically-enhanced contribution reads

$$D_{\mathrm{NLO}} = \frac{\alpha}{6\pi} \sum_M \frac{2\vec{v}_{0M} \cdot \vec{v}_{M0}}{D(E_M - E_0)} \left(\ln \frac{4(E_M - E_0)^2}{m_e^2} - \frac{1}{4} \right) \,, \tag{31}$$

with the coefficient of the logarithm,

$$\sum_M \frac{2\vec{v}_{0M} \cdot \vec{v}_{M0}}{D(E_M - E_0)} \equiv \frac{1}{m_e} \,, \tag{32}$$

being independent of any details of the bound state. To some extent, Eq. (32) simply reflects the fact that in D-dimension, the mass dimension of any normalized wave function equals $\frac{D}{2}$. Indeed, if one introduces the "dilatation operator"

$$\hat{D} = -i\vec{x} \cdot \vec{p} = -x^\mu \frac{\partial}{\partial x^\mu} \,, \tag{33}$$

then it is easy to see that Eq. (32) is equivalent to

$$\langle 0|\hat{D}|0\rangle = \frac{D}{2}\,, \tag{34}$$

nothing but the mass dimension of the wave function $\langle x|0\rangle$. This connection between the EMT form factor to the re-scaling property of the wave function is expected, since to some extent, T^{ii} also measures the response of the wave function under a spatial re-scaling $\vec{x} \to \lambda\vec{x}$.

Another important fact that should be mentioned is that this logarithm, although obtained through NRQED, matches precisely to the IR logarithms in relativistic QED for a free electron

$$D_{\mathrm{QED}} \sim \frac{\alpha}{6\pi m_e}\left(\ln\frac{4Q^2}{m_e^2} - \frac{11}{12}\right)\,, \tag{35}$$

providing a nice demonstration of the principle of the effective field theory. To some extent, in the presence of "criticality", in the sense that scale separations $\frac{\alpha m_e}{\alpha^2 m_e}$, $\frac{m_e}{\alpha m_e}$ become large, non-trivial structures with clean boundaries exists only near a small number of sharp peaks in the logarithmic scale. In the intermediate energy scales such as $\alpha m_e \ll \mu \ll m_e$, "colored noises", or self-similar random fluctuations without clear shape/boundary, characterized by simple scaling law possibly with logarithmic corrections, dominate. This vast sea of noise, although not splendid at first glance, actually serves as an amorphous "bridge" joining smoothly the otherwise divided worlds in the IR and UV, witnessing the "matching" between effective theories. If you look deeper into the cloud, you see beauties, such as the spin–spin correlator $G(z)$ for the two-dimensional Ising model as a function of separation z [24–26]

$$G(z) \to \frac{1}{z^{\frac{1}{4}}}\,, \tag{36}$$

or the running coupling constant $g(L)$ for the four-dimensional critical Ising model at scale L [27–29]

$$g(L) \to \frac{16\pi^2}{3\ln L}\,, \tag{37}$$

lasting forever.

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