TOWARDS A STABILITY ANALYSIS OF INHOMOGENEOUS PHASES IN THE QCD PHASE DIAGRAM*

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In these proceedings, we briefly show an alternative formalism to perform a stability analysis of inhomogeneous phases in QCD. We discuss how it is more general than the "classic" framework and some peculiarities with respect to its application to QCD-like theories.

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1. Introduction

The phase diagram of Quantum Chromo-Dynamics (QCD) is far from being fully understood. Amongst the several unorthodox phases being currently hypothesised, inhomogeneous phases belong to the ones most studied. Within *models* of the underlying theory, QCD, these phases are really very common. The most basic example is the Gross–Neveu (GN) model, a scalar contact interaction model of fermions in 1+1 dimensions. This is a good case study, given that much can be done analytically within the GN model, and a crystalline phase at large chemical potential, low temperatures does appear in the phase diagram [1]. One of the natural developments beyond the GN model would be, for instance, the Nambu–Jona-Lasinio (NJL) model. Usually, scalar, pseudoscalar, and sometimes vector and pseudovector contact interactions between fermions in $3 + 1$ dimensions are introduced [2].

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Another alternative model which retains some of the features of QCD and is used to model some of its physics is the Quark–Meson (QM) model [3, 4], which is based on meson-mediated interactions rather than contact interactions. Both within the NJL and QM models, these inhomogeneous phases were also found [5].

One work within the underlying theory, QCD, does exist [6]. Within the framework of Dyson–Schwinger Equations (DSE), they managed to find an inhomogeneous phase. Although no full confirmation that such a solution was more stable than the homogeneous ground state was attained, some necessary conditions for such were verified (for more details, see [7]). Thus, more and more confidence and excitement arose.

Some recent results, however, have dampened this excitement and certainly shaken the confidence that such inhomogeneous phases could exist in QCD. Lattice studies of the GN/NJL models with more than one spatial dimension [8–10] showed the region of the phase diagram where inhomogeneous phases are more stable to be highly regulator-scale-dependent, regulator-scheme-dependent, and in the limit where the regulator is taken to infinity, inhomogeneous phases are no longer found. This tension calls for more QCD-based studies and, in particular, some framework where the existence or in-existence of these phases can be guaranteed.

In these proceedings, we briefly review one of the methods used in model studies of inhomogeneous phases in QCD, namely, the stability analysis. We explain why such a framework is not suited to study inhomogeneous phases in QCD and show a novel method which is suited to tackle the issue.

2. Classical stability analysis

Take, for instance, an NJL model with scalar and pseudo-scalar interactions, in any dimension

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G\left\{\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi}i\gamma_5\vec{\tau}\psi\right)^2\right\} . \tag{1}$$

The "classical" method of performing a stability analysis goes as follows. We assume the condensates of the theory may be inhomogeneous, that is, the condensates can depend on the configuration space variable

$$\phi_{\mathrm{S}}(\boldsymbol{x}) = \left\langle \bar{\psi}(\boldsymbol{x})\psi(\boldsymbol{x})\right\rangle , \qquad \phi_{\mathrm{P}}(\boldsymbol{x}) = \left\langle \bar{\psi}(\boldsymbol{x})i\gamma_5\tau^3\psi(\boldsymbol{x})\right\rangle . \tag{2}$$

It is an elementary exercise in many-body quantum field theory to write the thermodynamic potential in the mean-field approximation (see, for instance, [11]). It reads as follows:

$$\Omega_{\mathrm{MF}} = -\frac{T}{V}\,\mathrm{Tr}\log\left(\frac{S^{-1}}{T}\right) + G\frac{1}{V}\int \mathrm{d}^3x\,\left(\phi_{\mathrm{S}}^2(\boldsymbol{x}) + \phi_{\mathrm{P}}^2(\boldsymbol{x})\right) . \tag{3}$$

We may then take the homogeneous solutions of the model and expand around them, considering only small fluctuations. We assume the condensates can be written as

$$\phi(\boldsymbol{x}) = \bar{\phi} + \delta\phi \,,$$

where $\bar{\phi}$ is the homogeneous condensate, obtained by solving the homogeneous gap equation. We can then expand the functional $\Omega_{\mathrm{MF}}[\phi]$ around the $\delta\phi = 0$ point. The zeroth order contribution is, naturally, the homogeneous thermodynamic potential $\Omega_{\mathrm{MF}}[\bar{\phi}]$, the first order always vanishes by the gap equation $\Omega^{(1)} = 0$, and the second order is the leading (non-trivial) order. For a detailed explanation of this procedure, the reader is referred to Ref. [12]. For our intents and purposes, it suffices to show the final result. The leading order contribution can be written as

$$\Omega^{(2)} = 2G^2 \sum_{\boldsymbol{q}_k} \left\{ \left|\delta\phi_{\mathrm{S},\boldsymbol{q}_k}\right|^2 D_{\mathrm{S}}^{-1}\left(\boldsymbol{q}_k^2\right) + \left|\delta\phi_{\mathrm{P},\boldsymbol{q}_k}\right|^2 D_{\mathrm{P}}^{-1}\left(\boldsymbol{q}_k^2\right) \right\} , \tag{4}$$

where D_{S}^{-1} and D_{P}^{-1} are the scalar and pseudoscalar inverse meson propagators calculated within the model, $\delta\phi_{\mathrm{S},\boldsymbol{q}_k}$ and $\delta\phi_{\mathrm{P},\boldsymbol{q}_k}$ are the Fourier modes of the condensate fluctuations with 3-momentum $\boldsymbol{q}_k$ (note that $\phi(x)$ is always taken to be static and x_0 independent). Therefore, by the mathematical structure of Eq. (4) we can see that, if and only if the inverse meson propagators become negative for any non-zero 3-momentum, the value of $\Omega^{(2)}$ can become negative and, thus, lower the free energy as compared with the homogeneous case.

The philosophy behind the stability analysis being as stated, we calculate a stability condition (negativity or positivity of which, depending on how it is defined, implies stable or unstable), which in this "classical" framework happens to be the inverse meson propagators. Therefore, we can easily calculate whether or not the homogeneous ground state, within these models, is unstable against the formation of small inhomogeneous condensates.

However, this is only possible given the two important conditions: (1) that the thermodynamic potential can be written as a functional on the condensates alone $\Omega[\phi]$, and (2) that the mathematical structure of $\Omega^{(2)}$ is compatible with writing a stability condition. Condition (2) might not be self-evident at this point, however, it is as crucial as condition (1). Had it not been possible to write $\Omega^{(2)}$ as something, namely, the stability condition (here, the inverse meson propagators) times a mod-squared — hence, strictly positive — contribution $|\delta\phi|^2$ of the fluctuations, it would not have been possible to perform a stability analysis, being completely agnostic with respect to the shape of $\delta\phi$. These conditions are in no way interdependent. It is easy to conceive that Ω could be written as a function of the potentials ϕ only, but $\Omega^{(2)}$ could depend on, say, $\delta\phi^2$ rather than $|\delta\phi|^2$ in which case, condition (1) is fulfilled and (2) is not. Similarly, we could conceive of an Ω which is not simply a functional of the condensates alone — it depends on other quantities as well — but at the end of the day, it *is* possible to write a moduation-shape agnostic stability condition somehow.

The point is, however, that conditions (1) and (2) above might be fulfilled by models of QCD, but neither of them applies to the underlying theory. Thus, if one intends to perform a stability analysis of the homogeneous ground state of dense quark–gluon matter, a new formalism is called for.

3. 2PI stability analysis

An equivalent analysis to the one outlined above goes as follows. Take a 2PI truncation of the effective action

$$\Gamma = \mathrm{Tr}\log\left[S^{-1}\right] - \mathrm{Tr}\left[\mathbf{1} - S_0^{-1}S\right] + \Phi_{\mathrm{2PI}}[S]\,, \tag{5}$$

where $\Phi_{\mathrm{2PI}}[S]$ is the sum of all 2PI diagrams. We find the Dyson–Schwinger equations [13] by

$$\frac{\delta\Gamma}{\delta S} = 0 \quad\Rightarrow\quad -S^{-1} + S_0^{-1} + \frac{\delta\Phi_{\mathrm{2PI}}}{\delta S} = 0 \quad\Rightarrow\quad \Sigma = \frac{\delta\Phi_{\mathrm{2PI}}}{\delta S}\,, \tag{6}$$

where $\Sigma = S^{-1} - S_0^{-1}$. In direct analogy with the previous formalism, we may propose that the *propagator*, rather than the condensates, be taken to be a homogeneous contribution plus a small inhomogeneous perturbation. Let the propagator be defined as

$$S\left(k_1, k_2\right) = \bar{S}\left(k_1\right)\delta\left(k_1 - k_2\right) + \delta S\left(k_1, k_2\right)\,, \tag{7}$$

where $\bar{S}(k)$ is the solution to the homogeneous Dyson–Schwinger Equations. Note that the first term contains a Dirac delta to ensure momentum conservation, as it is expected of the homogeneous case where translational symmetry is unbroken. A one-to-one correspondent analysis to the one outlined above can be derived and, here as well, the zeroth order is trivially the homogeneous effective action, the first order naturally vanishes by the gap equation[1]

$$\Gamma^{(1)} = \mathrm{Tr}\left[\overline{\frac{\delta\Gamma}{\delta S}}\delta S\right]\,,$$

since at the stationary point we demand $\delta\Gamma/\delta S$ to be zero as shown in Eq. (6). The second order is the leading order and the starting point to

[1] The overbar always indicates the quantity is evaluated at the homogeneous stationary point.

derive a stability condition is

$$\Gamma^{(2)} = \frac{1}{2}\,\mathrm{Tr}\left[\bar{S}^{-1}\left(k_1\right)\delta S\left(k_1,k_2\right)\bar{S}^{-1}\left(k_2\right)\delta S\left(k_2,k_1\right)\right] + \frac{1}{2}\,\mathrm{Tr}\left[\overline{\frac{\delta^2\Phi}{\delta S\delta S}}\delta S\delta S\right]. \tag{8}$$

The expression above is in a way equivalent to the $\Omega^{(2)}[\phi]$ described in the previous section. Since from the effective action we may calculate the thermodynamic potential by

$$\Omega(T,\mu) = -\frac{T}{V}\left(\Gamma(T,\mu) - \Gamma(0,0)\right), \tag{9}$$

we may use $\Gamma^{(2)}$ to try and write a stability condition.

However, it is certainly not this trivial. Not mentioning the more technical aspects — which are quite numerous — of calculating $\Gamma^{(2)}$, unfortunately, it is not possible to write a stability condition for non-local interaction functionals Φ such as in QCD. In the next subsection, we will discuss such issues and talk some about how they may be circumvented. Lastly, we show that, from the more 2PI $\Gamma^{(2)}$, we can recover the special case of $\Omega^{(2)}_{\mathrm{MF}}$ in NJL, thus showing the method described here is more general.

3.1. Comments on the interaction contribution to Γ^2

Take a simple truncation of Φ_{2PI} in QCD: a two-quark loop exchanging a non-dynamical (as in not self-consistently solved within the theory) gluon propagator $D^{ab}_{\mu\nu}(q)$ (this can be, for instance, the Maris–Tandy or Qin–Chang models [14, 15]). We have

$$\mathrm{Tr}\left[\overline{\frac{\delta^2\Phi_{\mathrm{2PI}}}{\delta S\delta S}}\delta S\delta S\right] = \mathrm{Tr}\left[\gamma_\mu t^a\delta S\left(k_1,k_2\right)\gamma_\nu t^b\delta S\left(k_2-q,k_1-q\right)D^{ab}_{\mu\nu}(q)\right].$$

It becomes clear that arriving at an expression like Eq. (4), where the inhomogeneous perturbation appears as a mod-squared and whatever term multiplies it can be taken as a stability condition, will not be possible due to the fact that the momentum dependence in δS gets shifted by the gluon momentum q.

We must conclude it is not possible to write a fully modulation shape agnostic stability condition in QCD. However, more modestly, only partially agnostic conditions can be derived. Take, for instance, the homogeneous part of the propagator $\bar{S}(k_1)\delta(k_1-k_2)$. In analogy to this, one thinks of writing the following. Assume the inhomogeneous part of the propagator is given by something like

$$\bar{S}(k_1)F(k_1-k_2)\,,$$

where F can be any function and when we take the limit $F(k) \to \delta(k)$, we recover the homogeneous case. However, this fails to abide by a fundamental property of the propagator, the adjoint relation

$$S\left(\omega_1, \vec{k}_1, \omega_2, \vec{k}_2\right)^\dagger = \gamma_4 S\left(-\omega_2, \vec{k}_2, -\omega_1, \vec{k}_1\right)\gamma_4 \,.$$

The following ansatz, though, works

$$\delta S\left(k_1, k_2\right) = \left(\bar{S}\left(k_1\right) + \bar{S}\left(k_2\right)\right) F\left(k_1 - k_2\right) \tag{10}$$

as long as $F(-k) = F(k)^\dagger$. This does conform to sufficient conditions to derive a stability condition. The terms $\delta S(k_1, k_2)$ and $\delta S(k_2 - q, k_1 - q)$ will contribute with factors of $F(k_1 - k_2)$ and $F(k_2 - k_1)$, respectively, which combine to $|F(k_1 - k_2)|^2$. Ansätze like these can yield stability conditions and they do retain some agnosticism, namely, the shape of F. However, suffice it to say, some ansatz had to be made.

3.2. The NJL case

We now show that the classic NJL result can be obtained from the 2PI formalism. However, in order to recover the one-to-one correspondent expression to Eq. (4), we must first bridge the following gap. The "classic" formalism shown above is based on an expansion of the condensates. Identically, since in the mean-field NJL, the self-energy is simply a linear combination of the condensates, we could see it as an expansion on the self-energy, *i.e.* $\Sigma = \bar{\Sigma} + \delta\Sigma$. In order to connect one approach to the other, we can write a Dyson series for the quark propagator

$$S = \bar{S} + \bar{S}\delta\Sigma S$$

and take both the self-energy and the propagator up to an inhomogeneous order one and obtain the relation[2]

$$\delta S = \bar{S}\delta\Sigma\bar{S} \,.$$

Now, take for instance a scalar NJL interaction potential

$$\Phi_{\mathrm{2PI}} = G\int\limits_{x,y} \delta(x-y)\,\mathrm{tr}\left[S(x,x)\right] \times \mathrm{tr}\left[S(y,y)\right] \,, \tag{11}$$

[2] One might think expanding δS to a quadratic order in $\delta\Sigma$ further terms would survive. This is in fact possible, however, higher-order terms happen to cancel out in this NJL calculation.

and, for example, let us assume scalar fluctuations, *i.e.* that $\delta\Sigma$ is also a Dirac scalar (contributions from other NJL vertexes as well as non-scalar fluctuations can be calculated by an analogous analysis). We then calculate $\Gamma^{(2)}$ from Eq. (8) and obtain the following relation[3]:

$$\begin{aligned}\Gamma^{(2)} &= \frac{1}{2}\int\limits_{p,q} |\delta\Sigma(q)|^2\,\mathrm{tr}\left[\bar{S}(p)\bar{S}(p-q)\right] \\ &\quad +G\int\limits_{p,k,q} |\delta\Sigma(q)|^2\,\mathrm{tr}\left[\bar{S}(p)\bar{S}(p-q)\right]\mathrm{tr}\left[\bar{S}(k)\bar{S}(k+q)\right] \\ &= \frac{1}{2}\int\limits_{p,q} |\delta\Sigma(q)|^2\left(\mathrm{tr}\left[\bar{S}(p)\bar{S}(p-q)\right]\times\underbrace{\left(1+2G\int\limits_{q}\mathrm{tr}\left[\bar{S}(k)\bar{S}(k+q)\right]\right)}_{D_{\mathrm{S}}^{-1}(q)}\right),\end{aligned} \tag{12}$$

where the term in the inner parenthesis *is* the inverse scalar meson propagator in the mean-field NJL (see Ref. [2]) and from Eq. (9), we can obtain the theromodynamic potential contribution via $\Omega^{(2)} = -T/V \times \Gamma^{(2)}$. We have an extra contribution of $\mathrm{tr}[\bar{S}(p)\bar{S}(p-q)]$ compared with Eq. (4). However, not only do we verify numerically it is always negative, we can argue that, if we take the interaction strength G to zero, this becomes the stability condition for a free Fermi gas with respect to the formation of crystalline condensates. Of course, the free Fermi gas is stable, thus, its $\Omega^{(2)}$ is always positive, *i.e.*, its $\Gamma^{(2)}$ is always negative. Therefore, with the current analysis, we recover the same stability condition of Eq. (4): negativity of the inverse meson propagator in mean-field NJL signifies an instability of the homogeneous state with respect to the formation of small inhomogeneous condensates.

4. Summary

As discussed in Introduction, the need for a final determination on whether or not inhomogeneous phases exist in the QCD phase diagram is pressing. In order to give such a statement, one needs to go beyond models of QCD. The classic framework for stability analysis is not applicable to theories where the interaction is non-local such as QCD. In fact, one would not be able to use the old formalism even for the beyond mean-field NJL and QM models.

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[3] Note that in the mean-field NJL the self-energy is local and thus we can take $\delta\Sigma(k_1,k_2) = \delta\Sigma(k_1-k_2)$.

We have developed a new basis for stability analysis which is valid for any field theory and one can use progressively better truncations of the Dyson–Schwinger Equations to approach the full QCD result.

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