CORNELL POTENTIAL FROM SOFT WALL HOLOGRAPHIC APPROACH TO QCD*

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We discuss the confinement potential of the Cornell type arising within the framework of the generalized Soft Wall holographic model to QCD. The generalized model includes a parameter controlling the intercept of the linear Regge spectrum. Our analysis shows that the Cornell potential obtained in the scalar channel leads to a quantitative consistency with the phenomenology and lattice simulations, while the agreement in the vector channel is only qualitative.

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1. Introduction

The quark confinement in strong interactions to this day remains an unsolved mystery and the heavy-quark potential is one of the key quantities that may help us to uncover it. In detailed lattice simulations (see, *e.g.*, the review [1]), it was demonstrated that its form is in agreement with the potential [2] of the Cornell type

$$V(r) = -\frac{\kappa}{r} + \sigma r + \text{const}\,. \tag{1}$$

This result imposes a serious restriction on viable phenomenological approaches modeling the dynamics of non-perturbative strong interactions: In the non-relativistic limit, they should be able to reproduce this behavior. One of the approaches that passes this check is the so-called Soft Wall (SW) holographic model [3, 4] as was demonstrated by Andreev and Zakharov in [5]. Due to the recent emergence of the generalized SW models in [6] which allows the arbitrary intercept in the linear radial Regge spectrum, it seems worthwhile to extend the analysis for the case of these models as

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well. In this paper, we review the general derivation procedure for the heavy-quark potential in holographic models, apply it to the case of the generalized SW model, and discuss some of the phenomenological consequences of this model.

2. Generalized Soft Wall holographic model

The generalized Soft Wall holographic model is defined by the 5D action [6, 7]

$$S = \int \mathrm{d}^4x\, \mathrm{d}z \sqrt{g}\, \mathrm{e}^{-cz^2} U^2\left(b, 0, \left|cz^2\right|\right) \mathcal{L}\,, \tag{2}$$

where U is the Tricomi hypergeometric function, $g = |\mathrm{det} g_{MN}|$, $\mathcal{L}$ is a Lagrangian density of some free fields in AdS_5. The metric is given by the standard Poincaré patch of the AdS_5 space with z playing the role of the fifth (called holographic) coordinate

$$g_{MN}\, \mathrm{d}x^M \mathrm{d}x^N = \frac{R^2}{z^2}\left(\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \mathrm{d}z^2\right)\,, \qquad z > 0\,. \tag{3}$$

In the case of vector 5D fields V_M, the 4D mass spectrum of this model takes the Regge form

$$m_n^2 = 4|c|(n+b)\,, \qquad n = 1, 2, \ldots \tag{4}$$

The model can be reformulated in such a way that the dilaton background is eliminated and instead, the AdS_5 metric is modified while keeping the resulting discrete spectrum untouched. For the standard SW model, such reformulation was proposed in [4]. It reads

$$g_{MN} = \mathrm{diag}\left(\frac{hR^2}{z^2}, \ldots, \frac{hR^2}{z^2}\right)\,, \tag{5}$$

where $h = \mathrm{e}^{-2cz^2}$. The generalized SW model can be also rewritten in a similar way by observing that if a 5D holographic model in its action contains a z-dependent background described by a function $B(z)$, then the action can be rewritten in the form of $S = \int \mathrm{d}^4x\, \mathrm{d}z \sqrt{\tilde{g}}\, \tilde{\mathcal{L}}$ with the modified metric [6], $\tilde{g}_{MN} = B^2 g_{MN}$. Substituting the background $B(z)$ in action (2), we obtain the following generalization for the function h in the modified metric (5):

$$h = \mathrm{e}^{-2cz^2} U^4\left(b, 0, \left|cz^2\right|\right)\,. \tag{6}$$

This background function serves as a starting point in the derivation of the potential within the generalized SW holographic model.

3. Holographic Wilson loop

The analytic and lattice calculations of the potential between static sources are usually based on the analysis of a Wilson loop. The holographic variant of this analysis was developed by Maldacena in Ref. [8]. Within the holographic framework, one considers a Wilson loop $\mathcal{L}$ placed in the 4D boundary of the 5D space with the time coordinate ranging from 0 to T and the remaining 3D spatial coordinates y from $-r/2$ to $r/2$. The expectation value of the loop in the limit of $T \to \infty$ is, as usual, $\langle W(\mathcal{C})\rangle \sim \mathrm{e}^{-TE(r)}$, where $E(r)$ is the energy of the quark–antiquark pair. Alternatively, this expectation value can be obtained via $\langle W(\mathcal{C})\rangle \sim \mathrm{e}^{-S}$, where S represents the area of a string world-sheet which produces the loop $\mathcal{L}$. Combining these two equations, one can compute the energy (the static potential) of configuration as $E = S/T$. The natural choice for the world-sheet area is the Nambu–Goto action with the modified Euclidean AdS metric (5). The background function $h(z)$ specifies a holographic model, the general requirement is that the metric must be asymptotically AdS at $z \to 0$. From the first integral of the NG action, which corresponds to the action's translational invariance, one can then obtain an integral expression for the distance r

$$r = 2\sqrt{\frac{\lambda}{c}} \int\limits_0^1 \mathrm{d}v\, \frac{h_0}{h} \frac{v^2}{\sqrt{1 - v^4 \frac{h_0^2}{h^2}}}\,, \tag{7}$$

where we introduced a new notation, $z_0 \equiv z|_{y=0}$, $h_0 \equiv h|_{z=z_0}$, $v \equiv \frac{z}{z_0}$, and $\lambda \equiv cz_0^2$. Then the expression for the energy takes the form

$$E = \frac{R^2}{\pi\alpha'}\sqrt{\frac{c}{\lambda}} \left[\int\limits_0^1 \frac{\mathrm{d}v}{v^2} \left(\frac{h}{\sqrt{1 - v^4 \frac{h_0^2}{h^2}}} - D \right) - D \right] . \tag{8}$$

Here, $D \equiv h|_{v=0}$ is the regularization constant arising from a certain regularization procedure. Full details of these derivations can be found, *e.g.*, in Ref. [9].

Consider now the case of the generalized vector SW holographic model. According to the discussions in Section 2, the background function is given by expression (6) which we will use to generalize the Andreev–Zakharov deformed AdS metric, *i.e.*, we will investigate the confinement properties of SW model with the following generalized background function ($c > 0$):

$$h = \mathrm{e}^{2cz^2} U^4\left(b, 0, cz^2\right) = \mathrm{e}^{2\lambda v^2} U^4\left(b, 0, \lambda v^2\right) . \tag{9}$$

Substituting (9) into the integrals for the distance in (7) and the energy in (8), we get

$$r = 2\sqrt{\frac{\lambda}{c}}\int_0^1 \mathrm{d}v \, \frac{U^4(b,0,\lambda)}{U^4\left(b,0,\lambda v^2\right)} \frac{v^2 \mathrm{e}^{2\lambda\left(1-v^2\right)}}{\sqrt{1 - v^4\,\mathrm{e}^{4\lambda(1-v^2)}\frac{U^8(b,0,\lambda)}{U^8(b,0,\lambda v^2)}}}\,, \tag{10}$$

$$E = \frac{R^2}{\pi\alpha'}\sqrt{\frac{c}{\lambda}}\left[\int_0^1 \frac{\mathrm{d}v}{v^2}\left(\frac{\mathrm{e}^{2\lambda v^2}U^4\left(b,0,\lambda v^2\right)}{\sqrt{1 - v^4\,\mathrm{e}^{4\lambda(1-v^2)}\frac{U^8(b,0,\lambda)}{U^8(b,0,\lambda v^2)}}} - D\right) - D\right], \tag{11}$$

where the regularization constant is $D = U^4(b,0,0)$. The last two expressions determine the energy as a function of the distance in the parametric form. For practical purposes usually the asymptotics of the potential at large and small distances are used. The derivation of these for the above integrals may be found, for instance, in [9].

Let us now briefly discuss the scalar case. In this case, the background function $h(z)$ of the metric in (9) is replaced by [6] (see also discussions in [9])

$$h = \mathrm{e}^{2cz^2/3}U^{4/3}\left(b,-1,cz^2\right) = \mathrm{e}^{2\lambda v^2/3}U^{4/3}\left(b,-1,\lambda v^2\right). \tag{12}$$

This changes the expressions for $r(\lambda)$ and $E(\lambda)$ to the following ones

$$r = 2\sqrt{\frac{\lambda}{c}}\int_0^1 \mathrm{d}v \, \frac{U^{4/3}(b,-1,\lambda)}{U^{4/3}\left(b,-1,\lambda v^2\right)} \frac{v^2 \mathrm{e}^{2\lambda\left(1-v^2\right)/3}}{\sqrt{1 - v^4\,\mathrm{e}^{4\lambda(1-v^2)/3}\frac{U^{8/3}(b,-1,\lambda)}{U^{8/3}(b,-1,\lambda v^2)}}}\,, \tag{13}$$

$$E = \frac{R^2}{\pi\alpha'}\sqrt{\frac{c}{\lambda}}\left[\int_0^1 \frac{\mathrm{d}v}{v^2}\left(\frac{\mathrm{e}^{2\lambda v^2/3}U^{4/3}\left(b,-1,\lambda v^2\right)}{\sqrt{1 - v^4\,\mathrm{e}^{4\lambda(1-v^2)/3}\frac{U^{8/3}(b,-1,\lambda)}{U^{8/3}(b,-1,\lambda v^2)}}} - D\right) - D\right], \tag{14}$$

with the regularization constant $D = U^{4/3}(b,-1,0)$.

4. Discussions

For a phenomenological analysis of our results, we should first provide a very brief review of the relevant phenomenology of the Cornell potential (1). In typical potential models for heavy quarkonia (for a review see, *e.g.*, [1]), the constant C is roughly $C \approx -0.3$ GeV. The potential in (1) was first proposed in Ref. [2] for a non-relativistic description of the charmonia spectrum. The linear confinement potential at large distances was inferred from lattice gauge theory and also inspired by the dual string model [10–13]. The spin averaged charmonia spectrum (plus ground states of bottomonia) with the inclusion of excited bottomonia, that probe the potential at smaller distances, results in parameter values [1], $\kappa \approx 0.51$ and $\sigma \approx 0.18\,\mathrm{GeV}^2$. The value of slope σ turns out to be remarkably insensitive to a chosen distance. The same value is typically used for the description of the light meson spectrum within the potential models.

After this brief reminder, we are ready to analyze the relevant phenomenological predictions of our generalized SW holographic model. Generally speaking, our predictions may depend on the choice of the normalization constant R^2/α'. Let us assume that this constant is b-dependent which can help us to amend the following theoretical discrepancy. The semiclassical quantization of hadron string of the Nambu–Goto type with tension σ and linearly growing with distance energy as in (1), leads to the slope of the angular and radial Regge mass spectrum $a = 2\pi\sigma$. On the other hand, the same slope in the SW holographic model is $a = 4c$, see, *e.g.*, the generalized SW spectrum (4). The physical values of σ and c remarkably agree with each other. Howerver, this agreement can be destroyed by the dependence $\sigma(b)$ as long as the parameter c in the SW model does not depend on b. Here, the normalization constant R^2/α' can restore the consistency: The actual σ in (1) is our $\sigma_\infty R^2/\alpha'$, hence, we may impose the consistency condition, $2\pi\sigma_\infty(b)R^2/\alpha' = 4c$, that should fix the normalization constant R^2/α'.

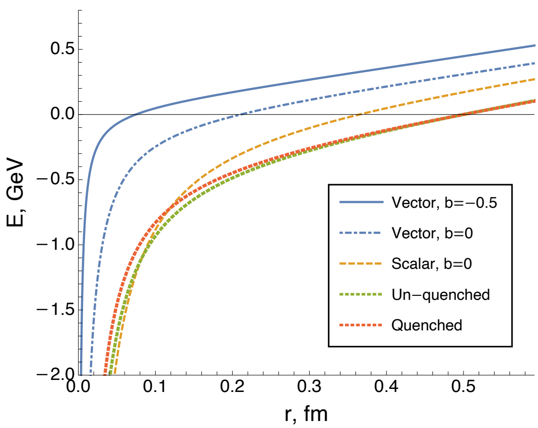

Fig. 1. The behavior of potential energy with distance for three examples of SW holographic model discussed in the text. The lattice data for the Wilson action SU(3) potential are taken from the review [1]. More precisely, we made use of the Cornell potential (1) that almost perfectly interpolates these data, the parameters of this potential are the following [1]: $\sigma = 0.18\,\mathrm{GeV}^2$, $\kappa = 0.295$ for the quenched approximation, and $\kappa \approx 0.36$ in the un-quenched case (sea quarks effects are taken into account), in both cases the constant C is fixed by the condition $E(0.5\text{ fm}) = 0$.

It is interesting to compare our models normalized by the above condition with the results of lattice simulations in the SU(3) gauge theory. In Fig. 1, we provide such a comparison for three typical cases: The vector SW model with $b = 0$ (the simplest standard variant), with $b = -0.5$, and the scalar SW model with $b = 0$ (the most consistent variant according to our analysis). One may see from Fig. 1 that the holographic confinement potential quantitatively compatible with the phenomenology arises in the scalar version of the SW holographic model, in which the physical value of intercept parameter b in the Regge-like spectrum seems to be close to zero.

5. Conclusions

Within the framework of the Soft Wall holographic model, the Cornell-type potential at large and short distances can be represented as a function of the intercept of the linear Regge spectrum both in the case of the vector and scalar backgrounds. Detailed comparisons with the phenomenology and lattice simulations demonstrate that the scalar background leads to a quantitative consistency with phenomenology, while the agreement in the vector case is qualitative only.

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