

GEOMETRICAL METHODS FOR THE ANALYTIC
EVALUATION OF MULTIPLE MELLIN–BARNES
INTEGRALS*

SUMIT BANIK

Physik-Institut, Universitat Zurich
Winterthurerstrasse 190, 8057 Zurich, Switzerland
and
Paul Scherrer Institut, 5232 Villigen PSI, Switzerland

SAMUEL FRIOT

Université Paris-Saclay, CNRS/IN2P3, IJCLab, 91405 Orsay, France
and
Université de Lyon, Université de Claude Bernard Lyon 1, CNRS/IN2P3
IP2I Lyon, UMR 5822, 69622 Villeurbanne, France

*Received 12 January 2024, accepted 16 January 2024,
published online 11 March 2024*

Two recently developed techniques of analytic evaluation of multifold Mellin–Barnes (MB) integrals are presented. Both approaches rest on the definition of geometrical objects conveniently associated with the MB integrands, which can then be used along with multivariate residues analysis to derive series representations of the MB integrals. The first method is based on introducing conic hulls and considering specific intersections of the latter, while the second one rests on point configurations and their regular triangulations. After a brief description of both methods, which have been automatized in the `MBConicHulls.wl` Mathematica package, we review some of their applications. In particular, we show how the conic hulls method was used to obtain the first analytic calculation of complicated Feynman integrals, such as the massless off-shell conformal hexagon and double-box. We then show that the triangulation method is even more efficient, as it allows one to compute these nontrivial objects and harder ones in a much faster way.

DOI:10.5506/APhysPolBSupp.17.2-A12

* Presented by S. Friot at the XLV International Conference of Theoretical Physics “Matter to the Deepest”, Ustroń, Poland, 17–22 September, 2023.

1. Introduction

This talk contribution is based on [1] and subsequent works [2–6], where the focus is on methods of analytic calculation of multiple Mellin–Barnes (MB) integrals, their automatization, and some of their possible applications. Such integrals are useful in many domains of physics and mathematics such as quantum field theory [7, 8], electromagnetic waves in turbulence [9], asymptotics [10], hypergeometric functions [11], *etc.* The need for a systematic method of analytic computation of MB integrals was recently emphasized for instance in [12], in the quantum field theory context, more specifically for the computation of Feynman integrals. Indeed, Feynman integrals are fundamental objects for scattering amplitudes in particle physics and one possible way to evaluate them is through their MB integral representations.

The `MBConicHulls.wl` *Mathematica* package, whose first version was made public in [1] and which we further developed in [2, 3], is an efficient tool for the analytic computation of MB integrals which rests on two original geometrical methods that we will present in this paper. In fact, prior to this work, there has been one attempt to build a *Mathematica* package dedicated to this task and called `MBsums.m`, which was published in the proceedings of the MTDD15 conference [13]. In the approach developed by these authors, the MB integrations are performed sequentially which, depending on their order, can in principle provide at the end different series representations of the MB integral. However, it does not seem that the results given by the `MBsums.m` package are always optimal, since when applied to the MB representations of complicated Feynman integrals, for instance, an explosion of the number of terms renders the obtained analytical expressions difficult to use. One may thus wonder if the iterative integration procedure is the right approach to follow. Moreover, the `MBsums.m` package, in its present version, is restricted to the evaluation of Mellin–Barnes integrals with straight contours of integration. Therefore, in addition to the above-mentioned motivations coming from [12], these two drawbacks of `MBsums.m` pushed us to revisit this problem and, inspired by the works of some mathematicians [14–16], our study gave birth to two non-iterative methods which can be used for the evaluation of multiple MB integrals with straight or non-straight contours, and which dramatically reduce the number of terms in the final expressions. These two new methods rest on the introduction, during the intermediate calculational steps, of geometrical objects associated with the MB integrand, and give the final results in terms of (multivariable) series representations of the hypergeometric type. The first method uses conic hulls, specific intersection of which yielding different series representations of the MB integral [1], while the second method is based on triangulations of some configurations of

points, each triangulation being in one-to-one correspondence with one series representation [3]. These two equivalent geometrical intermediate steps are followed by a multivariate residues analysis and, as already said above, the whole corresponding calculational procedures have been automated in the `MBConicHulls.wl` Mathematica package [17]. Although they give exactly the same final results, the triangulation method is much faster than the conic hulls method when used for the computation of complicated multifold MB integrals.

The outline of this contribution is as follows. In a first introductory part, we will recall what MB integrals are and give a short motivation for their study. In particular, their historical link to the hypergeometric functions theory where they can be used, among others, to study the transformation theory of the latter, will be briefly presented as well as their use as a tool for the computation of Feynman integrals¹. One will then switch to the main core of the paper which concerns the analytic evaluation of MB integrals. After summarizing known facts, we will present our two methods. Then, a brief description of some recent applications of the latter will be followed by our conclusions.

2. Evaluating multiple MB integrals analytically

The multifold MB integrals considered in this paper take the form

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} x_1^{z_1} \cdots x_N^{z_N} \frac{\prod_{i=1}^k \Gamma^{a_i}(\mathbf{e}_i \cdot \mathbf{z} + g_i)}{\prod_{j=1}^l \Gamma^{b_j}(\mathbf{f}_j \cdot \mathbf{z} + h_j)} \quad (1)$$

where a_i, b_j, k , and l are positive integers, $\mathbf{e}_i, \mathbf{f}_j$ are N -dimensional coefficient real vectors, $\mathbf{z} = (z_1, \dots, z_N)$, and g_i, h_j can be complex numbers. When not specified, as in Eq. (1), it is implicit that the contours of integrations are such that, for each of the gamma functions of the numerator of the MB integrand, they do not separate the corresponding set of poles in different subsets. This implies that, for some MB integrals, the contours cannot be straight lines parallel to the imaginary axis in the complex planes of integration.

The analytic evaluation of one-fold MB integrals is mathematically well-understood and fully described in books such as [10, 11]. The multifold case is much less tamed. Several theoretical physicists worked a lot on the elaboration of automatized tools necessary for the computation of MB representations of Feynman integrals (see, for instance, [18]) and, in this context,

¹ Due to the publication constraints of *Acta Physica Polonica B*, this part of the talk only appears in the `arXiv` version.

as mentioned previously in this paper, the `MBsums.m` *Mathematica* package of [13], which can be used for an iterative evaluation of multiple MB integrals with straight contours, was built a few years ago.

Nearly in parallel with these investigations, some mathematicians developed a non-iterative and rigorous approach based on the theory of multidimensional residues [19], in a series of papers [14–16]. However, to our knowledge, they did not explicitly work out the formulas necessary for the evaluation of N -fold MB integrals when $N > 2$, even in the non-logarithmic (or non-resonant) case². Only the $N = 2$ case is explicitly described in these papers. Inspired by these works and motivated to fill this gap, we developed a first alternative and efficient method [1] based on conic hulls which, although not established with the same level of rigor as in [14–16], has been tested on a considerable number of examples, either analytically when results were available for comparison, or numerically. We are now going to describe this method in more detail but for this, we have to first recall a few general facts about MB integrals.

There are two main types of MB integrals:

- The degenerate³ (also called “balanced”) type, where $\Delta \doteq \sum a_i \mathbf{e}_i - \sum b_j \mathbf{f}_j = \mathbf{0}$. In this case, several series representations of the corresponding MB integral coexist which are in general analytic continuations of one another. The MB representations of Feynman integrals belong to this class.
- The non-degenerate type, which satisfies $\Delta \neq \mathbf{0}$. In the latter case, one or more convergent series converge for all values of the scales. Additionally, asymptotic series also arise.

MB integrals can be further classified regarding the singularity structure of their integrand:

- The non-resonant (also called “generic”) case: Here, the number of singular hyper-planes intersecting at any pole is equal to the number of folds of the MB integral.
- The resonant (also called “logarithmic”) case, where the number of singular hyper-planes intersecting at some or all poles is greater than the number of folds of the MB integral, giving birth to logarithmic contributions in the series representations.

The conic hulls/triangulation methods described below can be used to compute MB integrals falling into any of these classes.

² The $N = 2$ logarithmic case was considered in [20] (for MB integrals with straight contours).

³ There is, in fact, another, slightly different, degenerate case, see [16].

2.1. Brief overview of the conic hulls method (non-resonant case)

The main steps of the conic hulls method, in the non-resonant situation, can be summarized as follows (we refer the reader to [1] for more details):

- **Step 1:** Find all possible N -combinations (where N stands for the number of folds of the MB integral) of the numerator gamma functions and retain non-singular ones.
- **Step 2:** Associate a series (building block) with each combination (obtained from a residue calculation at the poles of the corresponding gamma functions).
- **Step 3:** Construct a conic hull for each combination/series.
- **Step 4:** The largest intersecting subsets of conic hulls give series representations of the MB integral which simply are the sums of the corresponding building blocks.
- **Step 5:** The intersecting region gives the master conic hull. The latter is very useful to derive a master series, which can considerably simplify the convergence analysis of the series representation.

These computational steps have been automatized in a *Mathematica* package called `MBConicHulls.wl` [1, 17].

The resonant case is more complicated because it rests on a non-trivial multivariate residues analysis: the series representations cannot be simply reduced to sums of building blocks as in the non-resonant case (see [1] for more details). In order to handle the resonant case with the `MBConicHulls.wl` package, we internally used the `MultivariateResidues.m` *Mathematica* package [21].

At Step 3 above, for a given N -fold MB integral, the possible conic hulls are constructed from the $\mathbf{e}_i$ (of the gamma functions in the retained N -combinations) which play the role of basis vectors. Let us now focus on Step 4. For complicated MB integrals, there can be hundreds or even thousands of different series representations, each of these being a linear combination of many multivariable series selected in a large set of such objects. For instance, the MB representation of the off-shell massless conformal hexagon, which is a 9-fold MB integral [4, 12], has in the non-resonant case 194160 different series representations, all built from linear combinations of dozens of series that belong to a set of 2530 different series of 9 variables. Using other computational approaches of Feynman integrals, such as the negative dimension method [22, 23], the Yangian approach [12] or the method of bracket [24, 25], one would also find this set of 2530 series, however, in order to find the possible series representations built from the

latter (and in fact, even to find a single one ...), one would need to know the convergence regions of each of the 2530 series and study their possible intersections⁴, which obviously is a formidable, if not impossible task.

This is where the conic hulls method provides an important improvement compared to the alternative methods mentioned above: this very complicated convergence analysis is completely bypassed by the introduction of an appropriate set of conic hulls, as it is then sufficient to find their largest intersecting subsets in order to find the series representations of the MB integral. Moreover, as a bonus described in Step 5, each relevant intersection of conic hulls can be used to find a master conic hull, from which one can obtain the region of convergence (or a region included in the latter) of the series representation by finding the master series' ones mapped back from it. This is a conjecture that has been checked in many examples.

Let us note that, in its first version, the conic hulls method was developed for MB integrals with non-straight contours satisfying the usual property described in the beginning of this section. After applying it with success to the computation of difficult non-resonant and resonant MB representations of Feynman integrals [4, 5], we focused on the case of arbitrary straight contours of integration (parallel to the imaginary axis in the complex planes of the integration variables) that we solved in [2] and implemented in a new version of `MBConicHulls.wl`. The solution to this problem is simple: it is sufficient to use the generalized reflection formula

$$\Gamma(z - n) = \frac{\Gamma(z)\Gamma(1 - z)(-1)^n}{\Gamma(n + 1 - z)} \quad (2)$$

in such a way that the real part of each gamma function in the numerator of the MB integrand becomes positive along the contours. One can then apply the conic hulls method in the same way as in the non-straight contour case.

2.2. The triangulation method

The conic hulls method is very efficient, but still too slow when dealing with (very) complicated MB integrals. Trying to improve the method, we found that regular triangulations of particular point configurations are dual to the relevant intersections of conic hulls [3] and, therefore, that they can be used to find the different series representations of a given MB integral. The great advantage of triangulations is that, once automatized, they provide the results for complicated integrals in a much faster way than conic hulls⁵. We have thus upgraded the `MBConicHulls.wl` package in such a way that now, in addition to the original conic hulls analysis, it offers the possibility

⁴ Note that this would also be the case with the MB method developed in [9].

⁵ The computational times of the two methods are compared in [3].

to compute MB integrals from triangulations of point configurations. This is achieved by incorporating into the package the TOPCOM software [26] dedicated to triangulations. This way, more complicated MB integrals can be calculated analytically.

To apply the method, one needs to perform a change of the integration variables in order to rewrite Eq. (1) in the *canonical form*

$$I_N = \int_{-i\infty}^{+i\infty} \frac{dz_1}{2\pi i} \cdots \int_{-i\infty}^{+i\infty} \frac{dz_N}{2\pi i} \frac{\Gamma(-z_1) \cdots \Gamma(-z_N) \prod_{i=N+1}^{k'} \Gamma^{a'_i}(s'_i(z))}{\prod_{j=1}^l \Gamma^{b'_j}(t'_j(z))} x_1'^{z_1} \cdots x_N'^{z_N} \quad (3)$$

where we have pulled out the factors $\Gamma(-z_1) \cdots \Gamma(-z_N)$ in the numerator. One can then build the needed configuration of points as a set of N points

$$P_1 = e'_{i1}, \quad P_2 = e'_{i2}, \quad \cdots \quad P_N = e'_{iN} \quad (4)$$

and $\sum_{i=N+1}^{k'} a'_i$ additional points corresponding to the unit vectors of dimension $\sum_{i=N+1}^{k'} a'_i$. This set of $N + \sum_{i=N+1}^{k'} a'_i$ points is the one on which the triangulations are performed.

We applied the method on various Feynman integrals having MB representations with a high number of folds. Coming back to the hexagon and double-box [4], for instance, we could show that they have respectively 194160 and 243186 different series representations and, exploring these large sets, that there are simpler series representations than those obtained in [4] in the non-resonant case and in [1] in the resonant case. We also used the off-shell one-loop scalar massless N -point Feynman integral with generic powers of the propagators to test the computational possibilities of the triangulation approach. At $N = 15$ (the MB representation of the 15-point integral has 104 folds), TOPCOM seems to reach its limits but it is still possible to obtain some triangulations with the help of an average personal computer.

3. Conclusions

We have shown that N -fold MB integrals can be analytically computed in a non-iterative way by introducing conic hulls or triangulations of point configurations. Both the resonant and non-resonant cases can be handled by these methods which are automatized in the MBConicHulls.wl package. This package was used to perform the first analytic computation of the massless off-shell conformal hexagon and double box for generic and unit powers of the propagators. It can also be used to study the transformation theory of multivariable hypergeometric functions, which has been largely

unexplored, in a systematic way. However, a lot of work remains to be done for a full understanding of the field of investigations that these new approaches have opened. For instance, one would like to apply them to the calculation of MB representations of Feynman integrals having less scales than integration variables (presently, this can be done in certain cases but not all). It would also be interesting to better understand the master series conjecture and to find the rigorous link between triangulations and conic hulls. Another problem concerns the “white zones” which are regions where none of the series representations extracted from a direct calculation of many MB integrals converge: finding a systematic way to obtain series representations valid in these regions would be important. At last, a deeper study of non-degenerate MB integrals and their links with asymptotics in several variables, resurgence, *etc.* will probably yield interesting results.

S.F. thanks the organizers of MTDD23 for the very nice atmosphere of the conference and interesting talks and discussions.

REFERENCES

- [1] B. Ananthanarayan, S. Banik, S. Friot, S. Ghosh, *Phys. Rev. Lett.* **127**, 151601 (2021), [arXiv:2012.15108 \[hep-th\]](#).
- [2] S. Banik, S. Friot, *Phys. Rev. D* **107**, 016007 (2023), [arXiv:2212.11839 \[hep-ph\]](#).
- [3] S. Banik, S. Friot, [arXiv:2309.00409 \[hep-th\]](#).
- [4] B. Ananthanarayan, S. Banik, S. Friot, S. Ghosh, *Phys. Rev. D* **102**, 091901 (2020), [arXiv:2007.08360 \[hep-th\]](#).
- [5] B. Ananthanarayan, S. Banik, S. Friot, S. Ghosh, *Phys. Rev. D* **103**, 096008 (2021), [arXiv:2012.15646 \[hep-th\]](#).
- [6] S. Friot, G. Suchet-Bernard, [arXiv:2205.06247 \[math-ph\]](#).
- [7] V.A. Smirnov, «Analytic Tools for Feynman Integrals», Springer Tracts in Modern Physics, Vol. 250, *Springer Berlin, Heidelberg*, 2013.
- [8] I. Dubovyk, J. Gluza, G. Somogyi, «Mellin–Barnes Integrals: A Primer on Particle Physics Applications», Lecture Notes in Physics, Vol. 1008, *Springer Cham*, 2022, [arXiv:2211.13733 \[hep-ph\]](#).
- [9] R.J. Sasiela, «Electromagnetic Wave Propagation in Turbulence: Evaluation and Application of Mellin Transforms», Springer Series on Wave Phenomena, Vol. 18, *Springer-Verlag Berlin Heidelberg*, 1994.
- [10] R.B. Paris, D. Kaminski, «Asymptotics and Mellin–Barnes Integrals», in: «Encyclopedia of Mathematics and its Applications», Vol. 85, *Cambridge University Press*, Cambridge 2001.

- [11] O.I. Marichev, «Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables», Ellis Horwood Series in Mathematics and Its Applications, *Halsted Press*, 1983.
- [12] F. Loebbert, D. Müller, H. Münkler, *Phys. Rev. D* **101**, 066006 (2020), [arXiv:1912.05561 \[hep-th\]](#).
- [13] M. Ochman, T. Riemann, *Acta Phys. Pol. B* **46**, 2117 (2015), [arXiv:1511.01323 \[hep-ph\]](#).
- [14] M. Passare, A. Tsikh, O. Zhdanov, in: H. Skoda, J.-M. Trépreau (Eds.) «Contributions to Complex Analysis and Analytic Geometry», Aspects of Mathematics, Vol. E26, *Vieweg+Teubner Verlag*, Wiesbaden 1994, pp. 233–241.
- [15] M. Passare, A.K. Tsikh, A.A. Cheshel, *Teor. Math. Phys.* **109N3**, 381 (1996), [arXiv:hep-th/9609215](#).
- [16] A. Tsikh, O. Zhdanov, *Siberian Math. J.* **39**, 245 (1998).
- [17] MBConicHulls.wl webpage, <https://github.com/SumitBanikGit/MBConicHulls>
- [18] <https://mbtools.hepforge.org>
- [19] A. Tsikh, «Multidimensional Residues and Their Applications», *Nauka*, Novosibirsk 1988, [in Russian].
- [20] S. Friot, D. Greynat, *J. Math. Phys.* **53**, 023508 (2012), [arXiv:1107.0328 \[math-ph\]](#).
- [21] K.J. Larsen, R. Rietkerk, *Comput. Phys. Commun.* **222**, 250 (2018), [arXiv:1701.01040 \[hep-th\]](#).
- [22] I.G. Halliday, R.M. Ricotta, *Phys. Lett. B* **193**, 241 (1987).
- [23] G.V. Dunne, I.G. Halliday, *Phys. Lett. B* **193**, 247 (1987).
- [24] I. Gonzalez, *Nucl. Phys. B Proc. Suppl.* **205–206**, 141 (2010), [arXiv:1008.2148 \[hep-th\]](#).
- [25] B. Ananthanarayan, S. Banik, S. Friot, T. Pathak, *Phys. Rev. D* **108**, 085001 (2023), [arXiv:2112.09679 \[hep-th\]](#).
- [26] J. Rambau, in: A.M. Cohen, X.-S. Gao, N. Takayama (Eds.) «Proceedings of International Congress of Mathematical Software — ICMS 2002», *World Scientific*, 2002, pp. 330–340.