LIGHTCONE EXPANSION BEYOND LEADING POWER*

Sebastian Jaskiewicz

Albert Einstein Center for Fundamental Physics
Institut für Theoretische Physik, Universität Bern
3012 Bern, Switzerland

Received 15 January 2024, accepted 30 January 2024,
published online 11 March 2024

We discuss recent developments in descriptions of processes using power expansion around the lightcone within the Soft-Collinear Effective Theory. First, we present an overview of the systematically improvable framework that enables factorization of high-energy scattering processes beyond leading power in the expansion in ratios of energy scales. As an illustration of the relevant concepts, we describe the recently derived factorization theorem for the off-diagonal channel of the Drell–Yan production process at threshold. This example exposes endpoint divergences appearing in convolution integrals in factorization formulas. Lastly, we discuss the solution to these complications developed in the context of "gluon thrust" in e^+e^- collisions.

DOI:10.5506/APhysPolBSupp.17.2-A5

1. Introduction

Factorization theorems describing the decoupling of physical phenomena occurring at disparate energy scales and knowledge of universal objects governing singular limits of scattering processes are integral in enabling accurate theoretical predictions in collider physics. Perhaps the best-known example is the factorization of sufficiently inclusive scattering cross sections in hadronic collisions into perturbatively calculable short-distance part and parton distribution functions (PDFs), which capture the low-energy behaviour [1]. Despite the long history, most factorization theorems are formulated only at the leading power (LP) in the expansion in ratios of the disparate energy scales. Focusing on efforts utilising effective field theory methods, in this contribution, we discuss the recent advancements made towards descriptions valid at subleading powers.

* Presented at the XLV International Conference of Theoretical Physics "Matter to the Deepest", Ustroń, Poland, 17–22 September, 2023.

Studies of subleading power corrections are important for phenomenological applications [2, 3]. As has been shown recently for the case of inclusive Higgs production via gluon fusion, the size of the leading logarithmic (LL) corrections at next-to-leading power (NLP) can rival those of next-to-next-to-leading logarithmic (NNLL) at leading power [3]. Therefore, if towers of leading power logarithms are included to high logarithmic accuracy, the subleading power logarithms should also be included at least at the LL order to maintain control over the genuine size of the errors in the predictions.

On top of the phenomenological applications, investigations of power corrections present also an intriguing challenge from the theoretical perspective and plenty of work has recently been carried out on this topic in various contexts. A non-exhaustive list of recent studies includes investigations of Higgs production in gluon fusion at threshold, deep-inelastic scattering (DIS) at large Bjorken-x, hadronic e^+e^- annihilation, and the threshold Drell–Yan (DY) process [2–20]. Furthermore, studies have been carried out for the single Higgs boson production and decay amplitudes [21–34]. Advancements beyond leading power have also been achieved for variables such as N-jettiness [35–41], the q_{T} of the Higgs boson or the lepton pair [42–46], and in the context of QED [47–52], and B physics [53–57].

We begin by giving a brief overview of the general subleading power considerations within the Soft-Collinear Effective Theory (SCET) [58–62]. Namely, we discuss the basis of subleading power operators, the expanded Lagrangian with power suppressed soft-collinear interactions, and process specific kinematic power corrections. In order to illustrate the relevant concepts in a concrete example, we will focus on the recent derivation of next-to-leading power factorization formula for the quark–gluon channel of the Drell–Yan production process at threshold presented in [10]. The obtained results expose the presence of endpoint divergences, which is a ubiquitous complication appearing in various subleading power factorization theorems. In the last part of this contribution, we will discuss a solution to this problem in the context of "gluon thrust" in e^+e^- collisions using refactorization [16].

2. Sources of power corrections

In the following discussion, we adopt the subleading power SCET formalism developed in [63–68][1]. SCET describes the dynamics of soft and collinear partons. The collinear partons contain a large momentum component along one light-like direction and are suppressed along the remaining ones. It is convenient to use light-like reference vectors n_{i-}^{μ} and n_{i+}^{μ} satisfying $n_{i-} \cdot n_{i+} = 2$ and $n_{i-}^2 = n_{i+}^2 = 0$ for each of the collinear directions i.

[1] For an alternative approach of constructing power-suppressed operator basis in the label formulation of SCET, see [69–72].

In the first step, hard modes are integrated through a procedure which matches QCD to a basis of the SCET operators. The SCET operators are constructed out of collinear gauge-invariant building blocks [73]

$$\psi_i(x) \in \begin{cases} \chi_i(x) = W_i^\dagger(x)\xi_i(x) & i\text{-collinear quark}\,, \\ \mathcal{A}^\mu_{i\perp}(x) = W_i^\dagger(x)\big[iD^\mu_{i\perp}W_i(x)\big] & i\text{-collinear gluon}\,, \end{cases} \tag{2.1}$$

where $\xi_i(x) = \frac{\not{n}_{i-}\not{n}_{i+}}{4}\psi_i(x)$, $iD^\mu_i(x) = i\partial^\mu + g_s A^\mu_i(x)$, and the Wilson line is defined in (2.6) of [7]. Up to $\mathcal{O}(\lambda^2)$, a generic N-jet operator is [64]

$$J = \int \left[\prod_{ik} \mathrm{d}t_{i_k}\right] C(\{t_{i_k}\}) \prod_{i=1}^{N} J_i(t_{i_1}, t_{i_2}...)\,, \tag{2.2}$$

where $C(\{t_{i_k}\})$ is a generalised Wilson coefficient which captures the hard modes and J_i is a product of n_i collinear building blocks associated with a collinear direction n^μ_{i+}: $J_i(t_{i_1}, t_{i_2}...) = \prod_{k=1}^{n_i}\psi_{i_k}(t_{i_k}n_{i+})$. Each of the collinear building blocks in (2.1) has a scaling of $\mathcal{O}(\lambda)$ [61], where λ is the small power counting parameter of the theory and its specific form is dictated by the process under consideration. In the LP configuration, there is a single building block present in each of the collinear directions, as depicted in an N-jet example in panel (a) of figure 1. There are two ways to extend the basis of operators to subleading powers [64]. The first is through the introduction of $\partial^\mu_\perp$ derivatives which act on the building blocks already present at LP, bringing an $\mathcal{O}(\lambda)$ suppression. For example, $J^{A1}_i(t_i) = i\partial^\mu_{i\perp}\chi_i(t_i n_{i+})$ as shown in panel (b) of figure 1. Secondly, additional building blocks can be added within each collinear direction. Since the building blocks scale as λ, each insertion induces an $\mathcal{O}(\lambda)$ suppression. Examples of this are shown in panels (c) and (d) of figure 1, where $J^{B1}_i(t_{i_1}, t_{i_2}) = \mathcal{A}^\mu_{i\perp}(t_{i_1}n_{i+})\chi_i(t_{i_2}n_{i+})$

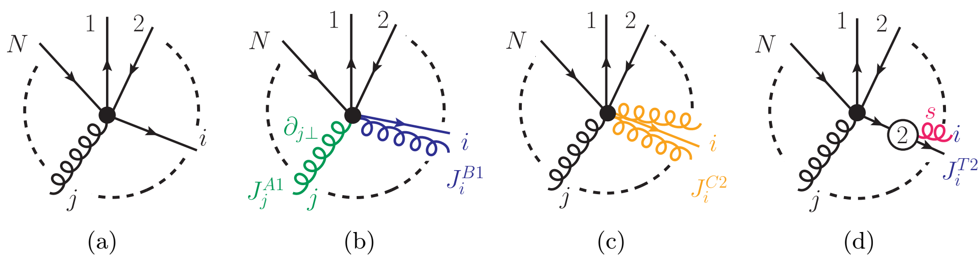

Fig. 1. Panel (a) shows a LP N-jet operator with one building block present in each of the N collinear directions, and panels (b), (c), and (d) depict possible power corrections, as described in the text.

and $J_i^{C2}(t_{i_1}, t_{i_2}, t_{i_3}) = \mathcal{A}^{\mu}_{i\perp}(t_{i_1} n_{i+})\mathcal{A}^{\mu}_{i\perp}(t_{i_2} n_{i+})\chi_i(t_{i_3} n_{i+})$. The basis is organised into currents J^{An}, J^{Bn}, $J^{Cn}, \ldots$ where the letters $A, B, C, \ldots$ denote the number of fields present in a particular collinear direction, and the number n gives the overall power suppression with respect to the LP operator in each sector. The sum of the power suppression from the different collinear sectors gives the overall power suppression for the N-jet operator. Next, we consider the SCET Lagrangian for QCD which is separated into N collinear parts and a global soft term, with Lagrangian terms in the collinear sectors are each systematically expanded in a small power counting parameter λ

$$\mathcal{L}_{\mathrm{SCET_I}} = \mathcal{L}_s + \sum_{i=1}^{N} \mathcal{L}_i \,, \qquad \mathcal{L}_i = \underbrace{\mathcal{L}_i^{(0)}}_{\mathcal{O}(\lambda^0)} + \underbrace{\mathcal{L}_i^{(1)}}_{\mathcal{O}(\lambda^1)} + \underbrace{\mathcal{L}_i^{(2)}}_{\mathcal{O}(\lambda^2)} + \ldots \tag{2.3}$$

The first term in the expansion, $\mathcal{L}_i^{(0)}$, is the LP contribution, and the remaining terms are the power corrections. The specific form of the Lagrangian expanded to $\mathcal{O}(\lambda^2)$ is given in [62]. Following [64], we adopt the interaction picture such that all operator matrix elements are evaluated with the LP SCET Hamiltonian, and the subleading power Lagrangian terms enter the basis as perturbations through time-ordered product insertions with lower power currents. At $\mathcal{O}(\lambda^{n+m})$, the time-ordered product operators take the form: $J_i^{T(n+m)}(t_i) = i\int \mathrm{d}^d x\, T\left\{J_i^{An}(t_i)\,\mathcal{L}_i^{(m)}(x)\right\}$. Lastly, the power suppression can enter via the so-called kinematic correction. This type of correction is process-specific and it originates in the phase-space approximations which are valid only up to LP. The kinematic correction is made up of appropriately expanding the phase space to subleading power accuracy. See [6] for an example in the case of DY production at threshold.

3. Threshold factorization of the Drell–Yan quark–gluon channel

Using the general construction outlined in the previous section, we now focus on the DY process at threshold and set the number of collinear directions $N = 2$. The threshold limit is characterised by $z \equiv Q^2/\hat{s} \to 1$, where Q^2 is the invariant mass of the final-state lepton pair, and $\hat{s}$ is the partonic centre-of-mass energy squared. Specifically, we consider the off-diagonal processes $g\bar{q}\,(qg) \to \gamma^* + X$. The derivation of the subleading power factorization theorem for this process has recently been presented in [10, 74]. In this contribution, we discuss important considerations and the key features. For details, we refer the interested reader to [10] where all the necessary ingredients to validate the NLP factorization to NNLO were computed.

We consider the invariant mass distribution for the Drell–Yan process

$$\frac{\mathrm{d}\sigma_{\mathrm{DY}}}{\mathrm{d}Q^2} = \sigma_0 \sum_{a,b} \int_{\tau}^{1} \frac{\mathrm{d}z}{z}\, \mathcal{L}_{ab}\left(\frac{\tau}{z}\right) \varDelta_{ab}(z) + \mathcal{O}\left(\frac{\varLambda}{Q}\right), \qquad \sigma_0 = \frac{4\pi\alpha_{\mathrm{em}}^2}{3N_{\mathrm{c}}Q^2 s}, \tag{3.1}$$

where $\mathcal{L}_{ab}(y)$ is the standard parton luminosity function, $\tau = Q^2/s$, and $\varLambda$ is the confinement scale of QCD. We seek to obtain the factorization theorem for the partonic part $\varDelta_{g\bar{q}}(z)$.

In the threshold limit, the final state contains only soft radiation and the hard matching to SCET operators is performed at amplitude level. In this limit, for the process $g\bar{q} \to \gamma^* + X$ to occur, the incoming gluon drawn from the PDF must be converted to a threshold collinear quark via the emission of a soft antiquark. The interaction between collinear fields and soft quarks is inherently a subleading power effect appearing for the first time in the SCET Lagrangian at $\mathcal{O}(\lambda)$ in $\mathcal{L}_{\xi q}^{(1)} = \bar{q}^{+} \not{\mathcal{A}}_{c\perp}^{(0)} \chi_c^{(0)}$, where the decoupling transformation [60] was performed and $\chi_c^{(0)}(z) = Y_+^{\dagger}(z_-)\chi_c(z)$, $\mathcal{A}_c^{(0)\mu}(z) = Y_+^{\dagger}(z_-)\mathcal{A}_c^{\mu}(z)Y_+(z_-)$, $q^{\pm} = Y_{\pm}^{\dagger}\, q_s$ with the soft Wilson lines defined in Eq. (2.4) of [7]. Hence, in contrast to the $q\bar{q}$-channel, the contribution due to the $g\bar{q}$-channel appears for the first time at NLP. It occurs via a time-ordered product insertion of $\mathcal{L}_{\xi q}^{(1)}$ with the LP current. The derivation of the factorization theorem proceeds as follows. The matching of the electromagnetic quark current to SCET fields is carried out at LP using

$$\bar{\psi}\gamma_\rho\psi(0) = \int \mathrm{d}t\, \mathrm{d}\bar{t}\, \widetilde{C}^{A0,A0}(t,\bar{t})\; J_\rho^{A0,A0}(t,\bar{t})\,, \tag{3.2}$$

where, *after* decoupling, in the notation for N-jet operators described above, we get

$$J_\rho^{A0,A0}(t,\bar{t}) = \bar{\chi}_{\bar{c}}(\bar{t}n_-)\, Y_-^{\dagger}(0)\gamma_{\perp\rho} Y_+(0)\chi_c(tn_+)\,. \tag{3.3}$$

We suppressed the superscript (0) on decoupled fields in the equation above. Power suppression is then generated via the aforementioned time-ordered product insertions of $\mathcal{L}_{\xi q}^{(1)}$. These insertions give rise to a new feature in subleading power factorization theorems with respect to their LP counterparts. Namely, the additional collinear fields introduced via time-ordered product insertions are too energetic to enter the final state in the threshold kinematics. Therefore, an amplitude level matching to PDF-collinear fields must first be performed giving rise to *NLP collinear functions* [6–8]. For the case of $g\bar{q}$-channel of DY, the matching is done onto the PDF-collinear gluon which gives rise to the following collinear matching equation [10, 74]:

$$i\int \mathrm{d}^d z\, T\left[\chi_{c,\gamma f}\left(tn_+\right)\mathcal{L}^{(1)}_{\xi q}(z)\right] = \frac{2\pi}{g_s}\int\frac{\mathrm{d}\omega}{2\pi}\int\frac{\mathrm{d}n_+p}{2\pi}\,\mathrm{e}^{-i(n_+p)t}\int\frac{\mathrm{d}n_+p_a}{2\pi}$$
$$\times\, G^{\eta,A}_{\xi q;\gamma\alpha,fa}\left(n_+p,n_+p_a;\omega\right)\hat{\mathcal{A}}^{\mathrm{PDF}\,A}_{c\perp\eta}(n_+p_a)\int \mathrm{d}z_-\,\mathrm{e}^{-i\omega z_-}\,\mathfrak{s}_{\xi q;\alpha,a}(z_-)\,. \quad (3.4)$$

In the above equation, η is a Lorentz index, α and γ are Dirac indices, and a, f, and A are fundamental and adjoint colour indices respectively. The soft structure, $\mathfrak{s}_{\xi q}$, originates from $\mathcal{L}^{(1)}_{\xi q}$. Making the indices explicit, it reads $\mathfrak{s}_{\xi q;\alpha,a}(z_-) = \frac{g_s}{in_-\partial_z}q^+_{\alpha,a}(z_-)$. After this step, the derivation of the factorization formula proceeds in the standard manner. The amplitude is squared, and the sum over the PDF-(anti)collinear state is carried out. The matrix element of the PDF-(anti) collinear fields is expressed in terms of the standard PDFs. The $g\bar{q}$ contribution to DY begins at NLP, so the phase space can be truncated at LP. The final NLP factorization formula can be simplified using generic properties of the collinear function as given in equation (2.30) of [10] which introduced a scalar version of the collinear function $G_{\xi q}(n_+p;\omega)$. After some further manipulations, we find [10]

$$\begin{aligned}\varDelta_{g\bar{q}}|_{\mathrm{NLP}}(z) \;=\; & 8H\left(Q^2\right)\int \mathrm{d}\omega\,\mathrm{d}\omega'\,G^*_{\xi q}\left(x_a n_+p_A;\omega'\right)\\ & \times G_{\xi q}(x_a n_+p_A;\omega)\,S\left(\varOmega,\omega,\omega'\right)\,. \end{aligned} \quad (3.5)$$

In this equation, $H(Q^2)$ is the well-known LP hard function, given by the square of the LP short-distance coefficient. The soft function $S(\varOmega,\omega,\omega')$ is given in (2.35) of [10]. A pictorial representation of the result in (3.5) is presented in figure 2. This concludes our overview of the derivation of the factorization theorem for the off-diagonal channel contribution to DY at NLP. Keeping the dimensional regulator in place, this result has been validated up to NNLO in [10] via an explicit computation of each of the ingredients appearing in the above result to the required perturbative accuracy: soft functions to $\mathcal{O}(\alpha_\mathrm{s}^2)$, collinear functions at $\mathcal{O}(\alpha_\mathrm{s})$.[2] The hard function is known up to three-loops [76]. Using the obtained results, it is also possible to expose the endpoint divergences appearing in this example. Namely, focusing on the collinear and soft piece, we have

$$\int_0^{\varOmega}\mathrm{d}\omega\,\underbrace{\left(n_+p\,\omega\right)^{-\epsilon}}_{\text{collinear piece}}\,\underbrace{\frac{1}{\omega^{1+\epsilon}}\frac{1}{(\varOmega-\omega)^{\epsilon}}}_{\text{soft piece}}\,. \quad (3.6)$$

[2] Equivalent collinear functions were calculated to $\mathcal{O}(\alpha_\mathrm{s}^2)$ for $H\to gg$ amplitude in [75].

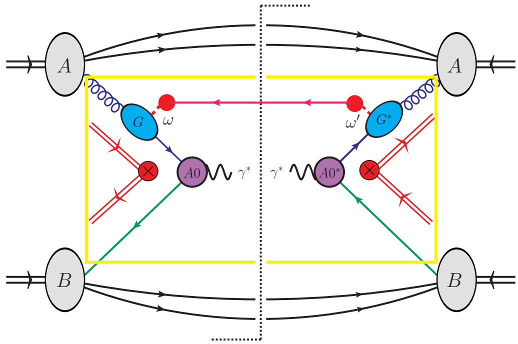

Fig. 2. Schematic representation of the factorization theorem in (3.5). The red lines represent soft fields, blue (green) lines represent (anti)collinear fields. The LP short-distance coefficient C^{A0} and its complex conjugate are the purple circles labelled "$A0$" and "$A0^*$", respectively. NLP collinear functions are represented by the blue ovals denoted by G, G^*. Lastly, ω and ω' are the convolution variables between the soft and collinear functions.

It is apparent that the integral is well defined when exact ϵ dependence is retained in the integrand. However, if the soft and collinear pieces are first expanded individually in the context of resummation, it is clear that a divergent integral is encountered. Namely, in this case, we find a term proportional to $\int \mathrm{d}\omega \delta(\omega) \ln(\omega)$. Therefore, the standard renormalization procedure and four-dimensional convolutions do not, in general, yield the correct structure of the NLP logarithms of $(1-z)$. In the $g\bar{q}$-channel, the issue arises already at the LL level, whereas it appears for the first time at NLL in the diagonal channels [7]. As mentioned in the introduction, the appearance of endpoint divergences in subleading power factorization theorems is a ubiquitous issue studied in a variety of contexts [11, 14, 16, 26, 30–32, 47, 55–57, 77]. In the next section, we discuss a solution for this problem using refactorization ideas developed for "gluon thrust" [16].

4. Refactorization in "gluon thrust"

We now switch focus to e^+e^- collisions and consider the "gluon thrust" event shape where at leading order, the gluon recoils a quark–antiquark pair

$$e^+e^- \to \gamma^* \to [g]_c + [q\bar{q}]_{\bar{c}}\,. \tag{4.1}$$

The thrust variable T is defined as [78, 79]: $T = \max_{\vec{n}} \frac{\sum_i |\vec{p_i}\cdot\vec{n}|}{\sum_i |\vec{p_i}|}$, where the index i runs over the final-state hadrons (partons). In the limit $\tau = 1 - T \to 0$, back-to-back jets are formed by the partons, and large logarithms in the τ variable appear at all orders in α_s. The quark–antiquark two-jet process contributing at LP in the τ expansion is known to high logarithmic accuracy [80]. Comparatively, much less is understood about the process in Eq. (4.1) beginning at NLP in the τ expansion with the leading term $\alpha_s \ln \tau$.

At the first order in α_s, the gluon jet can be induced in two ways: (I) The large anti-collinear momentum can be carried away by both the quark and the anti-quark, which creates a single jet that recoils against the collinear gluon. (II) The collinear gluon momentum is balanced by either the anti-collinear quark or anti-quark, which renders the remaining fermion soft.

From the SCET point of view, in situation (I), the $q\bar{q}g$ state is directly produced by hard scattering operator, see Section 2. Due to the subleading operator involved, we refer to this situation as the B-type scenario and it is depicted in the left diagram of figure 3. In this case, momentum conservation fixes only the total momentum of the $q\bar{q}$ pair. The amplitude depends on the fraction of the anti-collinear momentum carried by each particle. If one of these fractions becomes small, the corresponding parton becomes soft and must instead be counted as possibility (II), also referred to as the A-type scenario. In this contribution, the hard scattering vertex is the LP current and the full momentum of the quark (or anti-quark) is then transferred to the gluon, which renders the daughter fermion soft. Following the discussion in Section 2, this situation is captured by a time-ordered product insertion of $\mathcal{L}^{(1)}_{\xi q}$ which describes soft (anti-)quark emission. The A-type scenario is shown in the middle and right diagrams of figure 3.

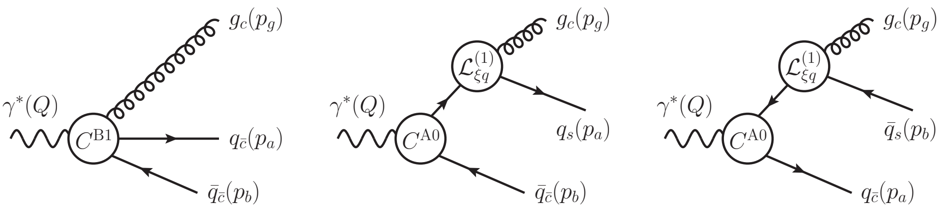

Fig. 3. SCET representation of the "gluon thrust" amplitude.

We begin with the factorization theorem for two-hemisphere invariant mass distribution of "gluon thrust" in Laplace space

$$\frac{1}{\sigma_0}\frac{\widetilde{\mathrm{d}\sigma}}{\mathrm{d}s_{\mathrm{R}}\mathrm{d}s_{\mathrm{L}}} = \int \mathrm{d}\omega \mathrm{d}\omega' \, \left|C^{A0}\right|^2 \times \mathcal{J}^{(\bar{q})}_{\bar{c}} \times \mathcal{J}_c\left(\omega,\omega'\right) \otimes S_{\mathrm{NLP}}\left(\omega,\omega'\right)$$
$$+ \int \mathrm{d}r\mathrm{d}r'\, C^{B1}(r) C^{B1}\left(r'\right)^* \otimes \mathcal{J}^{q\bar{q}}_{\bar{c}}\left(r,r'\right) \times \mathcal{J}^{(g)}_c \times S^{(g)} \,. \tag{4.2}$$

C denotes the hard matching coefficients, $\mathcal{J}$ stands for the jet functions, and S for the soft functions. Only the dependence on convolution variables which give rise to divergent integrals is retained. The r, r' convolution integrals diverge logarithmically for $r, r' \to 0, 1$, and the ω,ω' convolutions for $\omega, \omega' \to \infty$.

The soft quark from scenario (II) is contained in $S_{\rm NLP}$. In the overlap region, soft momentum ω carried by the soft quark is actually *large* and could count as part of $\mathcal{J}_{\bar{c}}^{q\bar{q}}$, with a *small* anti-collinear momentum fraction r. Removing this quark from $S_{\rm NLP}$ reduces it to $S^{(g)}$. Hence, taking all these changes into account, the hard process effectively changes from $A0$-type to $B1$-type. In these limits, we can identify $r = \omega/Q$, $r' = \omega'/Q$, such that the *integrands* of the two terms in (4.2) become identical. We can then perform a rearrangement at the *integrand* level, such that both terms are individually finite. From this stage, standard RG techniques can be employed to resum the logarithms in the hard, jet, and soft functions. Details of this procedure are presented in [16] which we summarize below.

The rearrangement of endpoint-singular terms is achieved by the introduction of the scaleless integral

$$\frac{2C_F}{Q}\, f(\epsilon)\, |C^{\rm A0}|^2 \widetilde{\mathcal{J}}_{\bar{c}}^{(\bar{q})} \widetilde{\mathcal{J}}_{c}^{(g)} \int_0^\infty d\omega d\omega' \,\frac{D^{\rm B1}}{\omega}\frac{D^{\rm B1*}}{\omega'} \left[\!\left[\widetilde{S}_{\rm NLP}\left(s_R, s_L, \omega, \omega'\right)\right]\!\right], \quad (4.3)$$

where we have suppressed the arguments in the D^{B1} functions defined in Eq. (57) of [16]. This integral is split into two terms $I_{1,2}$, $I_1 + I_2 = 0$, with I_1 defined by ω *or* ω' smaller than a parameter Λ and I_2 as the complement region, as depicted on the left-hand side of figure 4.

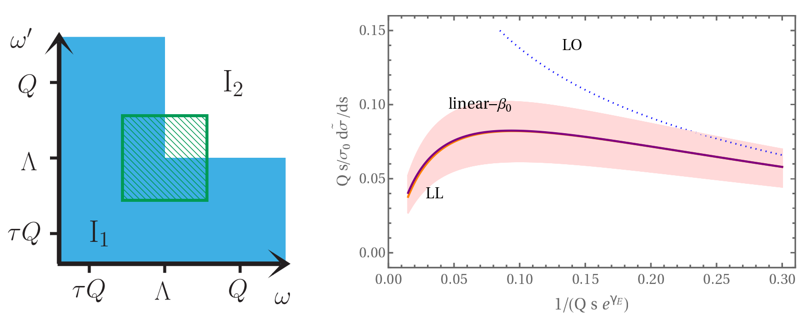

Fig. 4. The left panel shows the split of (4.3) into $I_1 + I_2$ in the ω–ω' plane as described in the text below Eq. (4.3). The right panel, displays the Laplace-space gluon thrust distribution at LL. The light red band is created by the variation of the initial scales as described in the text.

The endpoint rearrangement entails subtracting I_1 from the B-type contribution and I_2 from the A-type. The resulting expressions are separately endpoint-finite. The dependence on Λ cancels exactly between the two terms if no further approximations are made. The RG equations for the objects appearing in the factorization formulas can now be solved as presented in detail in Sections 5.1 and 5.2 of [16]. The end result for the leading logarithmic accurate resummed expression in the Laplace space is

$$\frac{1}{\sigma_0}\frac{\widetilde{\mathrm{d}\sigma}}{\mathrm{d}s_{\mathrm{R}}\mathrm{d}s_{\mathrm{L}}}\Big|_{\mathrm{LL}} = 2\cdot\frac{2C_F}{Qs_{\mathrm{R}}}\frac{\alpha_{\mathrm{s}}(\mu_c)}{4\pi}\ \exp\left[4C_F S\left(\mu_h,\mu_{\bar{c}}\right)+4C_A S\left(\mu_s,\mu_c\right)\right]$$
$$\times\left(\frac{1}{s_{\mathrm{L}}s_{\mathrm{R}}\mathrm{e}^{2\gamma_E}\mu_s^2}\right)^{-2C_A A(\mu_s,\mu_c)}\int\limits_{\sigma}^{Q}\frac{\mathrm{d}\omega}{\omega}\ \left(\frac{\omega}{s_{\mathrm{R}}\mathrm{e}^{\gamma_E}\mu_{s\Lambda}^2}\right)^{-2(C_F-C_A)A(\mu_{s\Lambda},\mu_{h\Lambda})}$$
$$\times\left(\frac{Q^2}{\mu_h^2}\right)^{-2C_F A(\mu_h,\mu_{\bar{c}})}\mathrm{e}^{[4(C_F-C_A)S(\mu_{s\Lambda},\mu_{h\Lambda})]}\left(s_R\mathrm{e}^{\gamma_E}Q\right)^{2C_F A(\mu_{h\Lambda},\mu_{\bar{c}})+2C_A A(\mu_c,\mu_{h\Lambda})}\ . \tag{4.4}$$

The functions $S\left(\nu,\mu\right)$ and $A_{\gamma_i}\left(\nu,\mu\right)$ are defined in [81]. To study the importance of NLL corrections, we vary the matching scales around the values adopted in (4.4). Three pairs of scales are varied $(\mu_h,\mu_{h\Lambda})$, $(\mu_c,\mu_{\bar{c}})$, $(\mu_s,\mu_{s\Lambda})$ by a factor of 1/2 and 2. Scale variation is computed by taking the resulting minimum and maximum values. The effect of this procedure is shown for $\frac{Qs}{\sigma_0}\frac{\widetilde{\mathrm{d}\sigma}}{\mathrm{d}s}$ in the right-hand panel of figure 4 as the light red band around the red curve (LL) which represents (4.4). We also display the LO and linear-β_0 truncation of the LL expression for comparison. The sizeable scale variation emphasizes the necessity of resummation at the NLL order. The endpoint-rearranged factorization formula derived in [16] provides the starting point for this systematically improvable analysis.

5. Summary

In this contribution, we highlighted the recent progress made towards achieving descriptions of factorization of physical scattering processes valid beyond leading power in the expansion in ratios of energy scales. In Section 2, we presented an overview of the SCET formalism which enables the systematic inclusion of subleading power corrections in these processes. In Section 3, we discussed the relevant concepts based on a concrete example of the factorization formula recently derived for the off-diagonal channel of the DY process at threshold. Using the relevant results, we discussed the appearance of endpoint divergences, and finally, we reviewed the recently developed solution in the context of "gluon thrust" in e^+e^- collisions. As is evident, studies of subleading power corrections present intriguing challenges from the theoretical perspective and are phenomenologically relevant for the upcoming precision era of the LHC and the Electron–Ion Collider.

REFERENCES

[1] J.C. Collins, D.E. Soper, G.F. Sterman, *Nucl. Phys. B* **261**, 104 (1985).

[2] M. Beneke *et al.*, *J. High Energy Phys.* **2020**, 094 (2020), arXiv:1910.12685 [hep-ph].

[3] M. van Beekveld *et al.*, *J. High Energy Phys.* **2021**, 114 (2021), arXiv:2101.07270 [hep-ph].

[4] D. Bonocore *et al.*, *J. High Energy Phys.* **2016**, 121 (2016), arXiv:1610.06842 [hep-ph].

[5] I. Moult *et al.*, *J. High Energy Phys.* **2018**, 013 (2018), arXiv:1804.04665 [hep-ph].

[6] M. Beneke *et al.*, *J. High Energy Phys.* **2019**, 043 (2019), arXiv:1809.10631 [hep-ph].

[7] M. Beneke *et al.*, *J. High Energy Phys.* **2020**, 078 (2020), arXiv:1912.01585 [hep-ph].

[8] I. Moult, I.W. Stewart, G. Vita, *J. High Energy Phys.* **2019**, 153 (2019), arXiv:1905.07411 [hep-ph].

[9] N. Bahjat-Abbas *et al.*, *J. High Energy Phys.* **2019**, 002 (2019), arXiv:1905.13710 [hep-ph].

[10] A. Broggio, S. Jaskiewicz, L. Vernazza, *J. High Energy Phys.* **2023**, 028 (2023), arXiv:2306.06037 [hep-ph].

[11] I. Moult *et al.*, *J. High Energy Phys.* **2020**, 089 (2020), arXiv:1910.14038 [hep-ph].

[12] I. Moult, G. Vita, K. Yan, *J. High Energy Phys.* **2020**, 005 (2020), arXiv:1912.02188 [hep-ph].

[13] A.H. Ajjath, P. Mukherjee, V. Ravindran, *Phys. Rev. D* **105**, 094035 (2022), arXiv:2006.06726 [hep-ph].

[14] M. Beneke *et al.*, *J. High Energy Phys.* **2020**, 196 (2020), arXiv:2008.04943 [hep-ph].

[15] A. Broggio, S. Jaskiewicz, L. Vernazza, *J. High Energy Phys.* **2021**, 061 (2021), arXiv:2107.07353 [hep-ph].

[16] M. Beneke *et al.*, *J. High Energy Phys.* **2022**, 144 (2022), arXiv:2205.04479 [hep-ph].

[17] R. van Bijleveld *et al.*, *J. High Energy Phys.* **2023**, 126 (2023), arXiv:2308.00230 [hep-ph].

[18] M. van Beekveld, L. Vernazza, C.D. White, *J. High Energy Phys.* **2021**, 087 (2021).

[19] M. Inglis-Whalen *et al.*, *Phys. Rev. D* **104**, 076018 (2021), arXiv:2105.09277 [hep-ph].

[20] A.H. Ajjath *et al.*, *J. High Energy Phys.* **2021**, 131 (2021), arXiv:2007.12214 [hep-ph].

[21] A.A. Penin, *Phys. Lett. B* **745**, 69 (2015), arXiv:1412.0671 [hep-ph]; *Erratum Phys. Lett. B* **771**, 633 (2017).

[22] T. Liu, A.A. Penin, N. Zerf, *Phys. Lett. B* **771**, 492 (2017), arXiv:1705.07910 [hep-ph].

[23] T. Liu, A.A. Penin, *Phys. Rev. Lett.* **119**, 262001 (2017), arXiv:1709.01092 [hep-ph].

[24] T. Liu, A.A. Penin, *J. High Energy Phys.* **2018**, 158 (2018), arXiv:1809.04950 [hep-ph].

[25] J. Wang, arXiv:1912.09920 [hep-ph].

[26] Z.L. Liu, M. Neubert, *J. High Energy Phys.* **2020**, 033 (2020), arXiv:1912.08818 [hep-ph].

[27] Z.L. Liu *et al.*, *J. High Energy Phys.* **2020**, 104 (2020), arXiv:2005.03013 [hep-ph].

[28] C. Anastasiou, A. Penin, *J. High Energy Phys.* **2020**, 195 (2020), arXiv:2004.03602 [hep-ph].

[29] T. Liu, S. Modi, A.A. Penin, *J. High Energy Phys.* **2022**, 170 (2022), arXiv:2111.01820 [hep-ph].

[30] Z.L. Liu *et al.*, *J. High Energy Phys.* **2023**, 183 (2023), arXiv:2212.10447 [hep-ph].

[31] Z.L. Liu *et al.*, *Phys. Rev. D* **104**, 014004 (2021), arXiv:2009.04456 [hep-ph].

[32] Z.L. Liu *et al.*, *J. High Energy Phys.* **2021**, 077 (2021), arXiv:2009.06779 [hep-ph].

[33] G.T. Bodwin *et al.*, *Phys. Rev. D* **104**, 016010 (2021), arXiv:2101.04872 [hep-ph].

[34] G.T. Bodwin *et al.*, *Phys. Rev. D* **104**, 116025 (2021), arXiv:2107.07941 [hep-ph].

[35] R. Boughezal, X. Liu, F. Petriello, *J. High Energy Phys.* **2017**, 160 (2017), arXiv:1612.02911 [hep-ph].

[36] I. Moult *et al.*, *Phys. Rev. D* **95**, 074023 (2017), arXiv:1612.00450 [hep-ph].

[37] I. Moult *et al.*, *Phys. Rev. D* **97**, 014013 (2018), arXiv:1710.03227 [hep-ph].

[38] M.A. Ebert *et al.*, *J. High Energy Phys.* **2018**, 084 (2018), arXiv:1807.10764 [hep-ph].

[39] R. Boughezal, A. Isgró, F. Petriello, *Phys. Rev. D* **97**, 076006 (2018).

[40] R. Boughezal, A. Isgró, F. Petriello, *Phys. Rev. D* **101**, 016005 (2020).

[41] G. Vita, arXiv:2401.03017 [hep-ph].

[42] M.A. Ebert *et al.*, *J. High Energy Phys.* **2019**, 123 (2019), arXiv:1812.08189 [hep-ph].

[43] L. Cieri, C. Oleari, M. Rocco, *Eur. Phys. J. C* **79**, 852 (2019), arXiv:1906.09044 [hep-ph].

[44] C. Oleari, M. Rocco, *Eur. Phys. J. C* **81**, 183 (2021), arXiv:2012.10538 [hep-ph].

[45] L. Buonocore *et al.*, arXiv:2311.12768 [hep-ph].

[46] G. Ferrera, W.-L. Ju, M. Schoenherr, arXiv:2312.14911 [hep-ph].

[47] G. Bell, P. Böer, T. Feldmann, *J. High Energy Phys.* **2022**, 183 (2022), arXiv:2205.06021 [hep-ph].

[48] E. Laenen *et al.*, *Phys. Rev. D* **103**, 034022 (2021), arXiv:2008.01736 [hep-ph].

[49] T. Engel, A. Signer, Y. Ulrich, *J. High Energy Phys.* **2022**, 097 (2022), arXiv:2112.07570 [hep-ph].

[50] T. Engel, *J. High Energy Phys.* **2023**, 177 (2023), arXiv:2304.11689 [hep-ph].

[51] T. Engel, arXiv:2311.17612 [hep-ph].

[52] R. Balsach, D. Bonocore, A. Kulesza, arXiv:2312.11386 [hep-ph].

[53] M. Beneke, C. Bobeth, R. Szafron, *Phys. Rev. Lett.* **120**, 011801 (2018).

[54] M. Beneke, C. Bobeth, R. Szafron, *J. High Energy Phys.* **2019**, 232 (2019), arXiv:1908.07011 [hep-ph].

[55] T. Feldmann *et al.*, *Phys. Rev. D* **107**, 013007 (2023), arXiv:2211.04209 [hep-ph].

[56] C. Cornella, M. König, M. Neubert, *Phys. Rev. D* **108**, L031502 (2023).

[57] T. Hurth, R. Szafron, *Nucl. Phys. B* **991**, 116200 (2023), arXiv:2301.01739 [hep-ph].

[58] C.W. Bauer, S. Fleming, M.E. Luke, *Phys. Rev. D* **63**, 014006 (2000).

[59] C.W. Bauer *et al.*, *Phys. Rev. D* **63**, 114020 (2001), arXiv:hep-ph/0011336.

[60] C.W. Bauer, D. Pirjol, I.W. Stewart, *Phys. Rev. D* **65**, 054022 (2002).

[61] M. Beneke *et al.*, *Nucl. Phys. B* **643**, 431 (2002), arXiv:hep-ph/0206152.

[62] M. Beneke, T. Feldmann, *Phys. Lett. B* **553**, 267 (2003), arXiv:hep-ph/0211358.

[63] M. Beneke *et al.*, *J. High Energy Phys.* **2005**, 071 (2005), arXiv:hep-ph/0411395.

[64] M. Beneke *et al.*, *J. High Energy Phys.* **2018**, 001 (2018), arXiv:1712.04416 [hep-ph].

[65] M. Beneke *et al.*, *J. High Energy Phys.* **2018**, 112 (2018), arXiv:1808.04742 [hep-ph].

[66] M. Beneke *et al.*, *J. High Energy Phys.* **2019**, 101 (2019), arXiv:1907.05463 [hep-ph].

[67] M. Beneke *et al.*, *PoS* **RADCOR2017**, 048 (2017), arXiv:1712.07462 [hep-ph].

[68] P. Böer, P. Hager, *J. High Energy Phys.* **2023**, 197 (2023), arXiv:2306.12412 [hep-ph].

[69] C. Marcantonini, I.W. Stewart, *Phys. Rev. D* **79**, 065028 (2009).

[70] D.W. Kolodrubetz, I. Moult, I.W. Stewart, *J. High Energy Phys.* **2016**, 139 (2016).

[71] I. Feige *et al.*, *J. High Energy Phys.* **2017**, 142 (2017), arXiv:1703.03411 [hep-ph].

[72] I. Moult, I.W. Stewart, G. Vita, *J. High Energy Phys.* **2017**, 067 (2017), arXiv:1703.03408 [hep-ph].

[73] R.J. Hill M. Neubert, *Nucl. Phys. B* **657**, 229 (2003), arXiv:hep-ph/0211018.

[74] S.E. Jaskiewicz, «Factorization and Resummation at Subleading Powers», Ph.D. Thesis, Technical University of Munich, 2021.

[75] Z.L. Liu *et al.*, *J. High Energy Phys.* **2022**, 075 (2022), arXiv:2112.00018 [hep-ph].

[76] T. Gehrmann *et al.*, *J. High Energy Phys.* **2010**, 094 (2010), arXiv:1004.3653 [hep-ph].

[77] R.F. del Castillo *et al.*, arXiv:2307.13025 [hep-ph].

[78] S. Brandt *et al.*, *Phys. Lett.* **12**, 57 (1964).

[79] E. Farhi, *Phys. Rev. Lett.* **39**, 1587 (1977).

[80] T. Becher M.D. Schwartz, *J. High Energy Phys.* **2008**, 034 (2008), arXiv:0803.0342 [hep-ph].

[81] M. Neubert, *Eur. Phys. J. C* **40**, 165 (2005), arXiv:hep-ph/0408179.