THE EMISSION OF SOFT-PHOTONS AND THE LBK THEOREM, REVISITED*

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Predictions for processes involving soft photons, up to next-to-leading power (NLP) in the photon energy, can be obtained using the Low–Burnett–Kroll (LBK) theorem. The consistency of the theorem has been a recent topic of investigation since it is traditionally formulated in terms of a non-radiative amplitude, which is evaluated with unphysical momenta. We address such questions and propose a formulation of the LBK theorem which relies on the evaluation of the non-radiative amplitude with on-shell, physical momenta. We use this form to numerically study the impact of NLP contributions to cross sections for pp and e^-e^+ processes involving soft-photon emission.

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1. Soft-photon anomaly

The theoretical framework of radiation in the low-energy (*i.e.* soft) limit is based on soft theorems, which enable the computation of radiative processes solely from the knowledge of the non-radiative amplitude and the external momenta. In QED, this universal factorization persists at next-to-leading power (NLP) in the soft-photon energy, as derived long ago by Low, Burnett, and Kroll (LBK) [1, 2]. Specifically, for an unpolarized cross section and keeping only the leading-power (LP) term in the soft expansion, the soft theorem relates the radiative amplitude $\mathcal{A}(p, k)$ and the non-radiative

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amplitude $\mathcal{H}(p)$ via

$$\overline{|\mathcal{A}(p,k)|}^2 = -\left(\sum_{ij=1}^{n} \eta_i\eta_j Q_i Q_j \frac{p_i\, p_j}{(p_i\, k)(p_j\, k)}\right) \overline{|\mathcal{H}(p)|}^2 \,, \tag{1}$$

where η_i is $+1(-1)$ for incoming (outgoing) particles.

However, the data gathered in several hadronic experiments [3–5] show a disagreement with the above formula, with an excess of photons that ranges between 4 and 8 times the theoretical predictions. Furthermore, there are plans to upgrade the ALICE detector [6, 7] with the aim of measuring ultra-soft photons. In light of this long-standing puzzle and proposed future measurements, further theoretical studies of soft-photon emissions are thus necessary. In particular, it is interesting to estimate the impact of NLP corrections to Eq. (1), as given by the LBK theorem.

2. Soft-photon emission via the LBK theorem

The diagrammatic derivation of the LBK theorem consists of discriminating the radiative amplitudes $\mathcal{A}^\mu_{\rm ext}$ and $\mathcal{A}^\mu_{\rm int}$, corresponding to radiation coming from the external lines and internal lines, respectively, as shown in Fig. 1 (b) and 1 (c). While the contribution from the former can be straightforwardly derived, internal radiation can be computed using gauge invariance, which yields

$$\mathcal{A} = \varepsilon_\mu \left(\mathcal{A}^\mu_{\rm ext} + \mathcal{A}^\mu_{\rm int}\right) \Longrightarrow k_\mu \left(\mathcal{A}^\mu_{\rm ext} + \mathcal{A}^\mu_{\rm int}\right) = 0 \,. \tag{2}$$

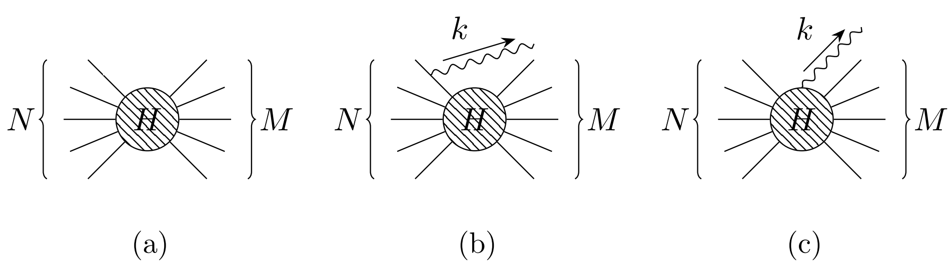

Fig. 1. Diagram (a) corresponds to the amplitude for a general non-radiative process, with N initial particles and M final ones. Diagram (b) corresponds to the photon radiation from an external line, and diagram (c) corresponds to the photon radiation from internal lines.

By exploiting this property, the LBK theorem at NLP can be easily derived and reads

$$\begin{aligned}\overline{|\mathcal{A}(p,k)|}^2_{\mathrm{LP+NLP}} &= -\sum_{i,j}\frac{(\eta_i Q_i p_i)\,(\eta_j Q_j p_j)}{(p_i\,k)(p_j\,k)}\overline{|\mathcal{H}(p)|}^2 \\ &\quad -\sum_{i,j}\eta_i Q_i Q_j \frac{p_{i\mu}}{(p_i\,k)} G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\overline{|\mathcal{H}(p)|}^2\,, \end{aligned} \tag{3}$$

where $G_j^{\mu\nu} = g^{\mu\nu} - \frac{p_j^\mu k^\nu}{p_j\,k}$. To derive Eq. (3), the amplitude is expanded in k while considering the other hard momenta p independent of k. Although this step is incompatible with the conservation of four-momentum, which can be stated as

$$\sum_i \eta_i p_i = k\,, \tag{4}$$

it is mathematically valid. However, the consequence is that the non-radiative amplitude in the right-hand side of Eq. (3) is evaluated using the momenta p, which are unphysical for this process, since $\sum \eta_i p_i \neq 0$. This might seem problematic because an amplitude is intrinsically defined for physical momenta, and it is not uniquely defined for unphysical momenta. Therefore, the value of $\mathcal{H}(p)$ is ambiguous, which translates into an ambiguity on $\mathcal{A}(p,k)$ and thus appears to invalidate Eq. (3). The argument, however, is not entirely correct, as shown in [8]. Indeed, although an ambiguity is present, it only affects the NNLP terms. More precisely, if we substitute $\overline{|\mathcal{H}|}^2 \to \overline{|\mathcal{H}|}^2 - \Delta$ in Eq. (3), the radiative amplitude changes according to

$$\overline{|\mathcal{A}|}^2 \to \overline{|\mathcal{A}|}^2 + \sum_{i,j}\frac{(\eta_i Q_i p_i)\,(\eta_j Q_j p_j)}{(p_i\,k)(p_j\,k)}\left[1+\eta_j\frac{(p_j\,k)p_{i\mu}}{p_i\,p_j}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\Delta(p)\,. \tag{5}$$

However, taking into account that the function $\Delta(p)$ must vanish when $\sum_i p_i = 0$, one can see that the ambiguity on $\overline{|\mathcal{A}|}^2$ is in fact of the order $\mathcal{O}(1)$ since

$$\begin{aligned}\delta\overline{|\mathcal{A}|}^2 &= \sum_{i,j}\frac{(\eta_i Q_i p_i)\,(\eta_j Q_j p_j)}{(p_i\,k)(p_j\,k)}\left[1+\eta_j\frac{(p_j\,k)p_{i\mu}}{p_i\,p_j}G_j^{\mu\nu}\frac{\partial}{\partial p_j^\nu}\right]\Delta(p) \\ &= \mathcal{O}(1)\,. \end{aligned} \tag{6}$$

The ambiguity is thus an NNLP effect. Therefore, the LBK theorem in the form shown in Eq. (3) is consistent and can provide reliable results.

Furthermore, due to the ambiguity in $\mathcal{H}$, Eq. (3) results in an entire family of equivalent formulations. Nevertheless, from the perspective of numerical implementations, it would be desirable to have a formulation that fulfils momentum conservation.

3. Shifted kinematics

There are multiple ways of restoring momentum conservation in Eq. (3) (see *e.g.* the discussion in [2] or, more recently, in [9, 10]). The approach we present here is based on the work described in [11, 12]. There, it was proved that the LBK theorem can be reformulated by shifting the momenta of the non-radiative amplitude. In this way, not only the conservation of four-momentum is restored, but also the derivatives of the non-radiative amplitude are removed, to get

$$\overline{|\mathcal{A}(p,k)|}^2 = -\left(\sum_{ij=1}^{n} \frac{(\eta_i Q_i p_i)\,(\eta_j Q_j p_j)}{(p_i\,k)(p_j\,k)}\right) \overline{|\mathcal{H}(p+\delta p)|}^2 \,, \tag{7}$$

where

$$\delta p_i^\mu = Q_i \left(\sum_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k\,p_l}{(p_k\,k)(p_l\,k)}\right)^{-1} \sum_j \left(\frac{\eta_j Q_j p_{j\nu}}{k\,p_j}\right) G_i^{\nu\mu} \,. \tag{8}$$

This formulation simplifies the implementation of the theorem, making it more suitable for numerical computations that use amplitudes generated by public tools. However, it turns out that the shifted momenta $p+\delta p$ in Eq. (7) are not on-shell, which can be problematic in some applications. Indeed, one finds

$$(p+\delta p)^2 = p^2 + Q_j^2 \left(\sum_{k,l} \eta_k \eta_l Q_k Q_l \frac{p_k\,p_l}{(p_k\,k)(p_l\,k)}\right)^{-1} = p^2 + \mathcal{O}\left(k^2\right) \,. \tag{9}$$

To solve this issue, one can modify the shifts in such a way that they fulfil momentum conservation and are on-shell to all orders in the soft-photon expansion, while still keeping Eq. (7) valid [8]. These modified shifts are defined as

$$\delta p_i^\mu = A Q_i \sum_j \frac{\eta_j Q_j}{k\,p_j} p_{j\nu} G_i^{\nu\mu} + \frac{1}{2} \frac{A^2 Q_i^2 \overline{|\mathcal{S}_{\mathrm{LP}}|}^2}{p_i\,k} k^\mu \,, \tag{10}$$

with

$$A = \frac{1}{\chi}\left(\sqrt{1-\frac{2\chi}{\overline{|\mathcal{S}_{\rm LP}|}^2}}-1\right), \qquad \chi = \sum_i \frac{\eta_i Q_i^2}{p_i\, k},$$
$$\overline{|\mathcal{S}_{\rm LP}|}^2 = -\left(\sum_{ij=1}^{n}\frac{(\eta_i Q_i p_i)\,(\eta_j Q_j p_j)}{(p_i\, k)(p_j\, k)}\right). \tag{11}$$

With these new modified shifts, it is possible to efficiently calculate NLP soft-photon emissions from arbitrary processes, and it is possible to use amplitudes numerically generated by public tools.

4. Numerical predictions for $\mu^-\mu^+\gamma$ production at e^-e^+ and pp collisions

Next, we study the numerical results obtained using three different versions of the LBK theorem, *i.e.* Eq. (3), Eqs. (7)+(8), and Eqs. (7)+(10). We do so by considering the $e^-e^+ \to \mu^-\mu^+\gamma$ process and comparing the results obtained with the LBK theorem to the exact results obtained without the soft-photon approximation, as shown in Fig. 2. One can see that the two formulations with shifts (called "NLP off-shell" and "NLP on-shell", respectively) seem to work better, at least for this process, but no clear difference is visible between the two formulations.

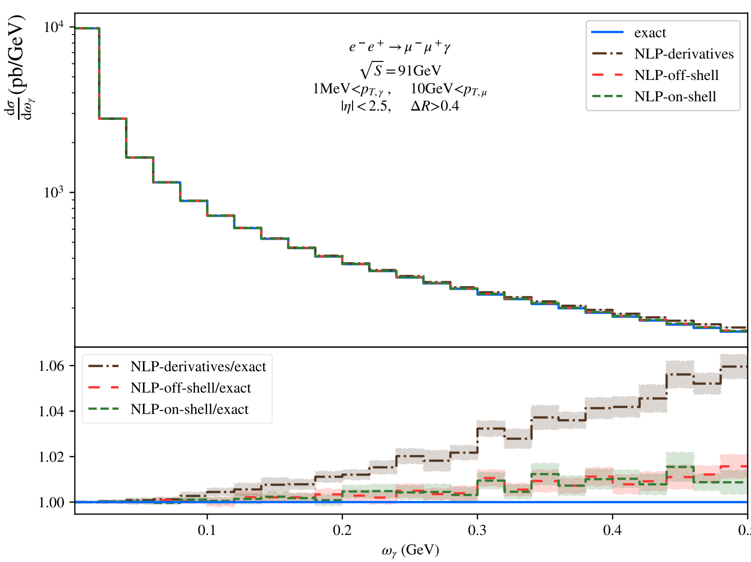

Fig. 2. Comparison of the soft-photon spectra between the different NLP formulations of the LBK theorem, normalised to the exact result (*i.e.* no soft-photon approximation).

Since the on-shell shifts of Eq. (10) enable writing the amplitudes generated numerically, we use this formulation of the LBK theorem for the remaining analyses shown in Figs. 3 and 4. Specifically, we compare the

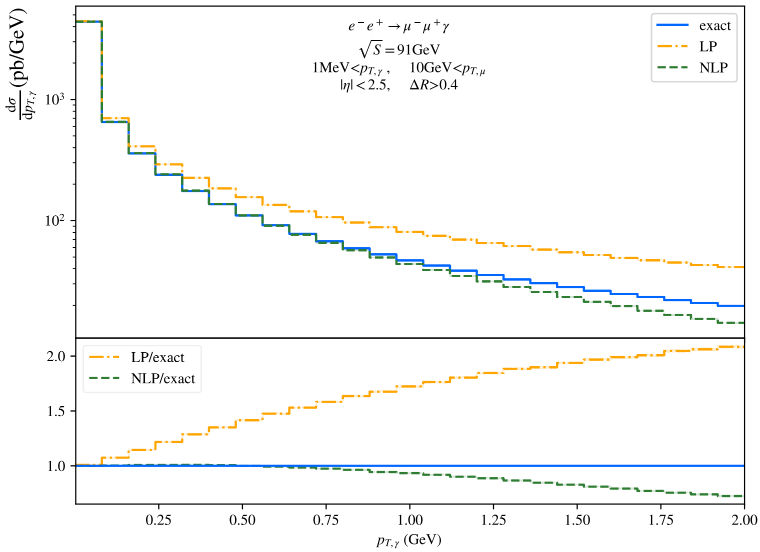

Fig. 3. Comparison of the p_{T} distribution calculated at the different accuracies in the soft expansion for e^+e^- collisions at $\sqrt{S} = 91$ GeV.

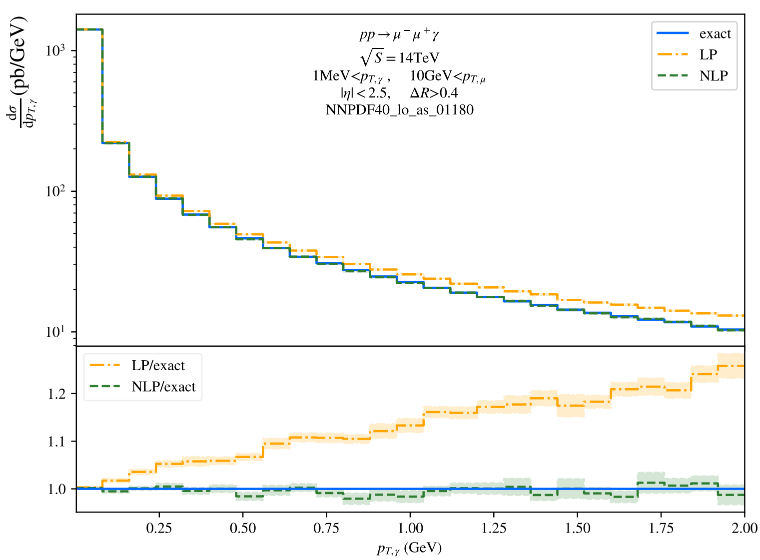

Fig. 4. Comparison of the p_{T} distribution calculated at the different accuracies in the soft expansion for pp collisions at $\sqrt{S} = 14$ TeV.

results for the $e^-e^+ \to \mu^-\mu^+\gamma$ and $pp \to \mu^-\mu^+\gamma$ processes, obtained using no approximation, with the corresponding expansions at LP and NLP. To do so, we apply the following kinematic cuts on the external particles: $p_{\mathrm{T},\gamma} > 1$ MeV, $p_{\mathrm{T},\mu} > 1$ MeV in the transverse momenta, $|\eta| < 2.5$ in the absolute pseudo-rapidity, and $\Delta R > 0.4$ for the angular distance between the particles. For both processes, we see that the NLP terms provide a very good approximation, with only a few per cent deviation for energies up to 1 GeV, and that the improvement with respect to the LP approximation is notable.

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