COMPUTING MULTI-LEG SCATTERING AMPLITUDES USING LIGHT-CONE ACTIONS*

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As is well known, computing multi-leg QCD scattering amplitudes using the standard elementary three- and four-particle vertices is cumbersome even at tree level, due to a number of diagrams growing dramatically with each external state. Over the last two decades, the problem was addressed in essentially two ways. The first approach uses the on-shell methods that try to eliminate fields as degrees of freedom whatsoever. The second approach is to construct a new field theory that contains new degrees of freedom that are more efficient in computing scattering amplitudes. One such example is the MHV action, which is based on the light-cone Yang–Mills action and where the new fields interact via multi-leg vertices related to the maximally-helicity-violating amplitudes. We discuss a further extension of such a theory, called the Z-field theory, which is obtained via the field transformation based on Wilson lines. Classically, it contains no triple couplings at all and thus is very efficient in computing tree amplitudes.

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1. Introduction

Our immediate intuition regarding scattering processes in Quantum Chromodynamics (QCD) usually involves quarks and gluons, considered 'elementary particles'. This notion of 'elementarity' is based, partially, on the space-time locality of the interactions of usual quark and Yang–Mills fields. Yet, the picture of a quark or a gluon as a well-defined particle is actually obscure. Not only quarks and gluons can emit undetectable collinear and soft gluons, but both types of fields are subject to color confinement; they are not the physical degrees of freedom of the QCD.

In the context of scattering amplitudes, the locality doctrine has been abandoned quite some time ago. The pioneering works by Witten [1] followed by the discovery of the MHV rules [2] and on-shell recursion relations [3] lead

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to an ‘S-matrix program’, where on-shell degrees of freedom are fundamental and allow for purely geometric treatment, while the off-shell fields and locality are inessential [4].

There are reasons, however, one may still need to use fields. Unlike in theories without confinement, in QCD, the on-shell scattering amplitudes are defined in the context of factorization theorems, that decouple, in a certain kinematic regime, the gauge-invariant non-perturbative hadronic part from scattering amplitudes. Although on-shell QCD degrees of freedom belong to the cohomology of the BRST transformation, the S-matrix of quarks and gluons is not physical. So far, proofs of factorization theorems use the ordinary field theory.

The idea of using on-shell degrees of freedom is very old and exists also in the context of field theory. The light-cone (or light-front) quantization [5–7] turns an off-shell line in an ordinary Feynman diagram into an on-shell line, at the price of non-local energy denominators. Such an approach can be also used to compute scattering amplitudes, see *e.g.* [8]. The essential feature of the light-front program is the quantization hyper-surface, which is defined by the constant light-cone time $x^+ = x^0 + x^3$ rather than the instant time x^0. It can be actually implemented even before the quantization — at the action level. The Lagrangian is defined at a constant light-cone time hypersurface and usually the light-cone gauge $A^+ = 0$ is implemented. Then it turns out that the $A^- = A^0 - A^3$ component can be removed via equations of motion (or integrated out from the partition function) leaving just the transverse degrees of freedom for the gauge fields [9]. We shall refer to such actions as *light-cone actions*.

There exists an interesting interplay between on-shell methods, locality, and light-cone actions, which can be best illustrated using the MHV rules. On the one hand, the original proposal [2] used the off-shell continuations of the MHV Parke–Taylor amplitudes [10], which soon was found to follow from a light-cone action, often called the MHV action [11]. The MHV action interaction vertices are local only in the light-cone time. On the other hand, the MHV rules were found to be an example of on-shell BCFW recursion [12]. This interplay is related to the twistor space formulation, where points correspond to non-local objects in Minkowski space and *vice versa*.

In the following contribution, we discuss three light-cone actions that classically correspond to the Yang–Mills theory, but implement very different degrees of freedom. In the next section, we review the ordinary Yang–Mills theory on the light-cone. Then, via a specific canonical field transformation, new collective degrees of freedom are introduced resulting in the MHV action. It turns out that the latter can be further transformed, so that the resulting light-cone action does not have any triple gluon interaction vertices and is thus very efficient in computation of scattering amplitudes.

2. Yang–Mills on the light-cone

Following the procedure outlined above, one can transform the standard Yang–Mills Lagrangian into the following, defined at fixed light-cone time x^+:

$$\mathcal{L}_{\mathrm{YM}}\left[A^\bullet, A^\star\right]\left(x^+\right) =$$
$$\int \mathrm{d}x^+ \int \mathrm{d}^3\boldsymbol{x} \Bigg\{ -\mathrm{Tr}\,\hat{A}^\bullet \Box \hat{A}^\star - 2ig\,\mathrm{Tr}\,\partial_-^{-1}\partial_\bullet \hat{A}^\bullet \left[\partial_- \hat{A}^\star, \hat{A}^\bullet\right]$$
$$-2ig\,\mathrm{Tr}\,\partial_-^{-1}\partial_\star \hat{A}^\star \left[\partial_- \hat{A}^\bullet, \hat{A}^\star\right] - 2g^2\,\mathrm{Tr}\left[\partial_- \hat{A}^\bullet, \hat{A}^\star\right]\partial_-^{-2}\left[\partial_- \hat{A}^\star, \hat{A}^\bullet\right]\Bigg\}, \quad (1)$$

where $\hat{A} = t^a A^a$ and the coordinates are defined as

$$v^+ = v_\mu \eta^\mu\,, \qquad v^- = v_\mu \widetilde{\eta}^\mu\,, \quad (2)$$
$$v^\bullet = v_\mu \varepsilon_\perp^{+\,\mu}\,, \qquad v^\star = v_\mu \varepsilon_\perp^{-\,\mu}\,, \quad (3)$$

with

$$\eta^\mu = \frac{1}{\sqrt{2}}(1,0,0,-1)\,, \qquad \widetilde{\eta}^\mu = \frac{1}{\sqrt{2}}(1,0,0,1)\,, \quad (4)$$
$$\varepsilon_\perp^{\pm\,\mu} = \frac{1}{\sqrt{2}}(0,1,\pm i,0)\,. \quad (5)$$

For convenience, we introduced a notation for three-vectors in position and momentum space in these coordinates

$$\boldsymbol{x} \equiv \left(x^-, x^\bullet, x^\star\right)\,, \qquad \boldsymbol{p} \equiv \left(p^+, p^\bullet, p^\star\right)\,. \quad (6)$$

The action for the Lagrangian (1) reads

$$S_{\mathrm{YM}}\left[A^\bullet, A^\star\right] = \int \mathrm{d}x^+ \mathcal{L}_{\mathrm{YM}}\left[A^\bullet, A^\star\right]\left(x^+\right)\,. \quad (7)$$

The transverse fields $A^\bullet$, $A^\star$ correspond to gluons with plus and minus helicity in the on-shell limit.

The action (7) has the following features:

— it is a scalar theory of two interacting fields, with the scalar propagator joining fields of opposite helicity,

— there are two triple point vertices, coupling $(++-)$ or $(--+)$ helicity fields,

— there is quartic vertex coupling $(++--)$ helicity fields; it can be demonstrated that it contains the instantaneous Coulomb interactions known in the light-front formalism.

The number of diagrams contributing to a scattering amplitude in the above theory is very large (of the same order as in the covariant formulation). This is simply a consequence of the presence of triple interaction vertices with mirror helicity configurations.

3. MHV action

In [11], it was observed that one can apply the canonical field transformation to the partition function of (7) to turn it into an action containing the MHV vertices. It was much later realized that the solution to the transformation equation is given by a path ordered exponential that can be thought of as the straight infinite Wilson line along a complex direction, with all directions integrated over [14] (see Fig. 1)

$$B_a^{\bullet}[A^{\bullet}]\,(x) = \int\limits_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{Tr}\left\{ \frac{1}{2\pi g} t^a \partial_- \,\mathbb{P}\exp\left[ig \int\limits_{-\infty}^{\infty} \mathrm{d}s\, \varepsilon_\alpha^+ \cdot \hat{A}\left(x + s\varepsilon_\alpha^+\right) \right] \right\}, \tag{8}$$

and [13]

$$B_a^{\star}[A^{\bullet}, A^{\star}](x) = \int \mathrm{d}^3\boldsymbol{y} \left[\frac{\partial_-^2(y)}{\partial_-^2(x)} \frac{\delta B_a^{\bullet}(x^+;\boldsymbol{x})}{\delta A_c^{\bullet}(x^+;\boldsymbol{y})} \right] A_c^{\star}(x^+;\boldsymbol{y})\,, \tag{9}$$

where

$$\varepsilon_\alpha^{\pm} = \epsilon_\perp^{\pm} - \alpha\eta\,. \tag{10}$$

The above vector (with the 'plus' transverse projection) sets the direction of the Wilson line through the slope parameter α. It is interesting that (10) has a form of a polarization vector, namely ε_α with $\alpha = p^{\bullet}/p^+$ being a proper polarization vector for momentum p. The vector ε_α^+ spans a plane in complex Minkowski space which we call a self-dual plane (the 'conjugate' vector ε_α^- defines the anti-self-dual plane). The name stems from the property that every tangent bivector defined as $u^\mu w^\nu - u^\nu w^\mu$ is self-dual. It can be checked that this property applies to the considered plane.

The form of the field transformations follows from two simple requirements: *(i)* the transformation is canonical to avoid the field-dependent Jacobian in the partition function, *(ii)* the kinetic term with one of the triple vertices is mapped to solely free term in the transformed action.

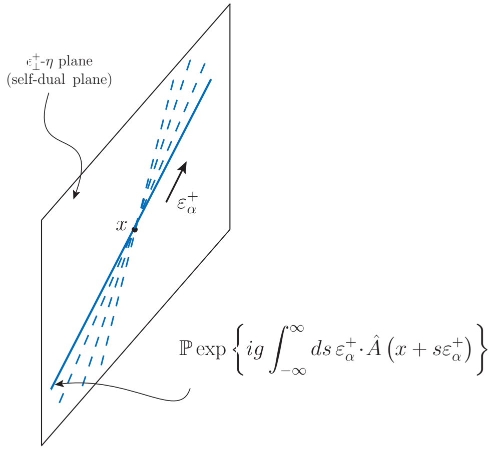

Fig. 1. The solution to the field transformation $A^\bullet \to B^\bullet$ is given by the straight infinite Wilson line lying on the self-dual plane. The direction of the line has to be integrated over, to get the field in the MHV action. Picture taken from [13].

Obtaining the solution to the field transformations in momentum space and substituting to the Yang–Mills action leads to the MHV action

$$S_{\mathrm{MHV}}\left[B^\bullet, B^\star\right] = \int \mathrm{d}x^+ \left(\mathcal{L}_{+-} + \mathcal{L}_{--+} + \dots + \mathcal{L}_{--+\dots+} + \dots\right) , \tag{11}$$

where

$$\begin{aligned}\mathcal{L}_{--+\dots+} \;&=\; \int \mathrm{d}^3\boldsymbol{p}_1 \dots \mathrm{d}^3\boldsymbol{p}_n \delta^3\left(\boldsymbol{p}_1 + \dots + \boldsymbol{p}_n\right) \tilde{\mathcal{V}}^{b_1\dots b_n}_{--+\dots+}\left(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\right) \\ &\times \tilde{B}^\star_{b_1}\left(x^+;\boldsymbol{p}_1\right) \tilde{B}^\star_{b_2}\left(x^+;\boldsymbol{p}_2\right) \tilde{B}^\bullet_{b_3}\left(x^+;\boldsymbol{p}_3\right) \dots \tilde{B}^\bullet_{b_n}\left(x^+;\boldsymbol{p}_n\right) ,\end{aligned} \tag{12}$$

with the MHV vertices

$$\begin{aligned}\tilde{\mathcal{V}}^{b_1\dots b_n}_{--+\dots+}\left(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n\right) \;&=\; g^{n-1}\left(\frac{p_1^+}{p_2^+}\right)^2 \mathrm{Tr}\left(t^{b_1}\dots t^{b_n}\right) \\ &\times \frac{\tilde{v}^{*4}_{21}}{\tilde{v}^*_{1n}\tilde{v}^*_{n(n-1)}\tilde{v}^*_{(n-1)(n-2)}\dots\tilde{v}^*_{21}} .\end{aligned} \tag{13}$$

Above, instead of the usual spinor products that would need a proper off-shell continuation, we have used the following symbols:

$$\tilde{v}_{ij} = p_i^+\left(\frac{p_j^\star}{p_j^+} - \frac{p_i^\star}{p_i^+}\right) , \qquad \tilde{v}^*_{ij} = p_i^+\left(\frac{p_j^\bullet}{p_j^+} - \frac{p_i^\bullet}{p_i^+}\right) \tag{14}$$

defined for four-momenta p_i, p_j not necessarily on-shell.

The MHV action has the following properties:

— The MHV vertex (13) is local in the light-cone time and holomorphic.

— The number of diagrams contributing to amplitudes is greatly reduced, as one of the triple gluon vertices was eliminated. There is still present the lowest triple MHV vertex though, which itself does not represent any physical amplitude in the on-shell limit.

4. The $\boldsymbol{Z}$-field action

The MHV theory was obtained by applying field transformations that account only for self-dual interactions. That is, the transformation resums the interactions due to the self-dual part given by the $(++-)$ triple gluon vertex. These interactions can be completely accommodated by the Wilson lines in the self-dual plane.

A natural question arises: Can we define a transformation that will do the same with the other triple gluon vertex? That is, can we resum both self-dual and anti-self-dual interactions via field transformations?

The answer is positive. In [15], a Wilson line-based transformation was found and a new action, called *Z-field action*, was derived. It is best to write the transformation in terms of a generating functional for canonical transformation

$$\mathcal{G}[A^{\bullet}, Z^{\star}](x^+) = -\int \mathrm{d}^3\boldsymbol{x} \;\; \mathrm{Tr}\, \hat{\mathcal{W}}^{-1}_{(-)}[Z](x)\, \partial_- \hat{\mathcal{W}}_{(+)}[A](x)\,, \tag{15}$$

where $\mathcal{W}^a_{(\pm)}[K](x)$ is a generic straight infinite Wilson line functional of a field $\hat{K}$ along the vector $\varepsilon^{\pm}_{\alpha}$

$$\mathcal{W}^a_{(\pm)}[K](x) = \int\limits_{-\infty}^{\infty} \mathrm{d}\alpha\, \mathrm{Tr}\left\{ \frac{1}{2\pi g} t^a \partial_-\, \mathbb{P}\exp\left[ig \int\limits_{-\infty}^{\infty} \mathrm{d}s\, \varepsilon^{\pm}_{\alpha} \cdot \hat{K}\left(x + s\varepsilon^{\pm}_{\alpha}\right)\right]\right\} . \tag{16}$$

Unlike for the MHV case, here we also have to use the Wilson line along ε^{-}_{α}, which defines an anti-self-dual plane.

Solving the transformations and substituting to the Yang–Mills action leads to the new action with the following general structure:

$$\begin{aligned} S\left[Z^{\bullet},Z^{\star}\right] \;=\; & \int \mathrm{d}x^{+}\Bigg\{ -\int \mathrm{d}^3\boldsymbol{x}\,\mathrm{Tr}\,\hat{Z}^{\bullet}\Box\hat{Z}^{\star} \\ & +\mathcal{L}_{--++}+\mathcal{L}_{--+++}+\mathcal{L}_{--++++}+\dots \\ & +\mathcal{L}_{---++}+\mathcal{L}_{---+++}+\mathcal{L}_{---++++}+\dots \\ & \vdots \\ & +\mathcal{L}_{---\dots-++}+\mathcal{L}_{---\dots-+++}+\mathcal{L}_{---\dots-++++}+\dots \Bigg\}, \end{aligned} \tag{17}$$

The second row represents the MHV vertices, the same as in the MHV theory, but without the triple vertex. Other vertices have only higher multiplicity and can be calculated — see [15]. Similar to the MHV case, they are local in the light cone time. We shall represent those vertices as a dot with multiple legs. The opposite helicity legs of the vertices are connected by a scalar propagator, as is evident from the action.

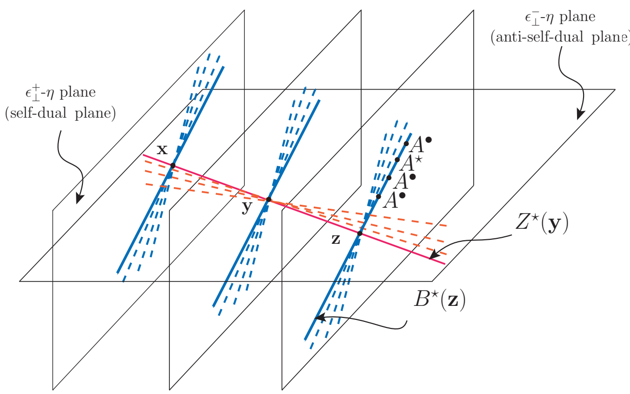

Fig. 2. The geometry of the $Z^{\star}$ field (situation for the $Z^{\bullet}$ is analogous). The vertical planes are self-dual planes, where MHV theory fields reside. The new $Z^{\star}$ field is a Wilson line of those fields, thus it is a 'Wilson line of Wilson lines'. Picture taken from [15].

In [15], authors have computed tree-level scattering helicity amplitudes up to 8 legs to verify this new way of computing amplitudes. The maximal number of diagrams was just 13, which is considerably less than any other action-based theory.

In [16], it was conjectured that the number of diagrams for the split helicity case (*i.e.* same helicity together) with $n+2$ plus helicity and $m+2$ minus helicity is given by the Delannoy number series

$$D(n,m) = \sum_{i=0}^{\min(n,m)} \binom{m}{i}\binom{n+m-i}{m} = \sum_{i=0}^{\min(n,m)} 2^i \binom{m}{i}\binom{n}{i}. \tag{18}$$

Those numbers give a number of paths in a 2D lattice, where just three moves are allowed: right, up, and up-right. The hypothesis is firmly based on the helicity content of the new theory.

In order to further verify the Z-field theory and the Delannoy number hypothesis in an engineering thesis [17] a complete computation of all 9-point amplitudes was carried out, which we outline below. It should be remarked that for this multiplicity of gluons, ordinary methods would give an unbearable number of diagrams.

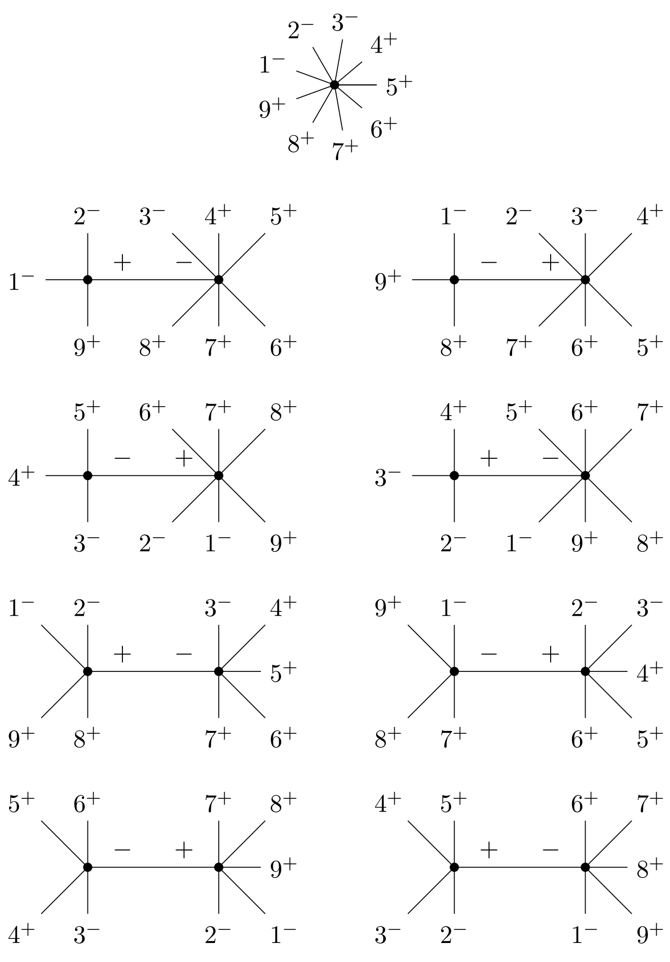

Fig. 3. All diagrams contributing to the NMHV 9 leg amplitude.

For the considered case of 9 external gluons, there are 6 non-trivial helicity configurations, as shown in Table 1. For each case, we identified all the contributing diagrams and found a match between the number of diagrams and the prediction of the Delannoy number series Eq. (18). As an example, consider the configuration $(---++++++)$ with $m=1$ and $n=4$. From Eq. (18), $D(4,1)=9$ which precisely matches the total number of 9 diagrams shown in Fig. 3. We checked numerically that the sum of the diagrams accompanied by the proper symmetry factors gives the correct results, see [17] for details. This work on amplitude with 9 gluons further led to the development of an algorithm that enables the identification of all diagrams for any number of gluons at tree level in a straightforward way.

Table 1. All helicity configurations in the case of 9 gluon amplitude.

Helicity configuration	Number of diagrams
$--+++++++$	1
$---++++++$	9
$----+++++$	25
$-----++++$	25
$------+++$	9
$-------++$	1

5. Summary

We have discussed three theories describing interactions of gluons related by a canonical field transformation. All the theories were formulated 'on the light-cone', so that the number of degrees of freedom was reduced. The three theories demonstrate — on the one hand — the growing complexity of the theory due to partially non-local interactions and infinite tower of vertices, and the amazing simplification in computing scattering amplitudes on the other. In particular, the Z-field theory does not have any triple gluon vertices.

Beyond the tree level, the situation gets much more complex. Due to an anomaly in the self-dual (or anti-self-dual) sector of the Yang–Mills theory, the same helicity amplitudes are non-zero at loop level. They cannot be obtained from the classical Z-field action alone. In [18], we have developed a method that in [16] was applied to account for the missing contributions. The quantum corrections to the Z-field theory are discussed in a separate contribution.

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