PRECISION PHYSICS AT HIGH-ENERGY COLLIDERS AND LOW-ENERGY CONNECTIONS*

Janusz Gluza, Biswajit Karmakar

Institute of Physics, University of Silesia, Katowice, Poland

Received 14 April 2024, accepted 3 June 2024,
published online 5 August 2024

Theory and high-energy physics connected with present and future colliders need progress in precision low-energy studies. In particular, at the FCC-ee collider, a better knowledge of input parameters by roughly one order of magnitude is required for many quantities, starting with $\alpha_{\mathrm{QED}}, \alpha_{\mathrm{s}}$, m_W, m_Z, m_H, m_t. We focus on $\alpha_{\mathrm{QED}}(M_Z^2)$ where analysis indicates that the uncertainty determination for this parameter should be roughly five times better than the presently determined value. In turn, better knowledge of the low-energy non-perturbative hadronic contribution to α_{QED} from $\Delta\alpha_{\mathrm{had}}$ with the five light quarks is required. In this context, we mention recent low-energy tensions concerning pion production processes which contribute to $e^+e^- \to \gamma^* \to$ hadrons, so $\Delta\alpha_{\mathrm{had}}$. We also discuss another important high–low energy connection: the influence of lepton flavor violating intensity frontier processes with CP heavy neutrino effects on the non-standard processes at high-energy colliders.

DOI:10.5506/APhysPolBSupp.17.5-A3

1. Introduction

One of the main achievements of the XX century particle physics and science was the discovery of the running of gauge (electromagnetic and strong) coupling constants. The discovery goes beyond atomic physics and chemistry where the fine structure constant is *par excellence* constant [1]

$$\begin{aligned}\alpha_{\mathrm{QED}} &= e^2/4\pi\epsilon_0\hbar c \\ &= 7.297\,352\,5693(11)\times 10^{-3} = 1/137.035\,999\,084(21)\ [0.15\ \mathrm{ppb}]\,. \quad (1)\end{aligned}$$

The dependence of coupling constants on energy is also one of the main arguments for the concept of gauge couplings merging in Grand Unified Theories (GUT). Below the GUT energy scale, high-energy collider physics

* Presented by J. Gluza at the 30$^{\mathrm{th}}$ Cracow Epiphany Conference on *Precision Physics at High Energy Colliders*, Cracow, Poland, 8–12 January, 2024.

requires an improvement in the determination of $\alpha_{\rm QED}$ at the weak scale, notable at the Z resonance region, which is one of the FCC-ee key modes [2]. As indicated in Eq. (1), $\alpha_{\rm QED}$ in the classical limit is known very precisely. It complies with other SM input parameters, where $\frac{\delta G_\mu}{G_\mu} \sim 8.6 \times 10^{-6}$ and $\frac{\delta M_Z}{M_Z} \sim 2.4 \times 10^{-5}$. On the other hand, in the Z-resonance, $1/\alpha(M_Z^2) = 128.953(016)$ [3] differs by 6% from $\alpha(0)$. More importantly, precision falls by about five orders mainly due to the non-perturbative hadronic effects, contributing about half of the amount to $\alpha(M_Z^2)$. At present, $\frac{\delta\alpha(M_Z^2)}{\alpha(M_Z^2)} \sim 0.9 \div 1.6 \times 10^{-4}$, while FCC-ee/CEPC/ILC [2, 4, 5] demand $\frac{\delta\alpha(M_Z^2)}{\alpha(M_Z^2)} \sim 5 \times 10^{-5}$. For the detailed discussion on the $\alpha_{\rm QED}$ errors budget, see [3].

In the next section, we will briefly describe the present status of $\alpha_{\rm QED}$ in the context of relevant hadron (pion) processes, which are in the domain of low-energy e^+e^- studies.

In the second part of this work, we focus on non-standard lepton flavor violating (LFV) effects at high-energy colliders and how low-energy processes are connected with them and help in precise predictions. After all, LFV processes are precision physics, where feeble, non-zero effects are investigated and searched for. Here, we stress the role of combined CP charge and parity discrete symmetries of heavy neutrinos, which decide about magnitudes of interference effects for heavy neutrinos in LFV amplitudes.

2. Impact of low-energy e^+e^- hadronic data on $\alpha(M_Z^2)$

Determining precisely even the simplest four-fermion, vector-boson, and Higgs-boson production and decay processes require very precise input parameters where higher-order quantum corrections are needed. Defining weak mixing angle θ by axial and vector couplings for charged Q_f fermions $\sin^2 \Theta_f = \frac{1}{4|Q_f|}\left(1 - \frac{v_f}{a_f}\right)$, quantum corrections from gauge boson self-energies, vertices, and boxes are gathered in the Δr_f parameter

$$\sin^2 \Theta_f \, \cos^2 \Theta_f = \frac{\pi\, \alpha}{\sqrt{2}\, G_\mu\, M_Z^2} \, \frac{1}{1 - \Delta r_f}\,, \tag{2}$$

$$\Delta r_f(\alpha, G_\mu, M_Z, m_H, m_{f\neq t}, m_t) = \Delta\alpha - f\left(\sin^2 \Theta_f\right)\, \Delta\rho + \Delta r_{f\,\rm sublead}\,. \tag{3}$$

$\Delta\rho$ accounts for WW, ZZ and $\gamma - Z$ self-energy terms, $\Delta\rho = \frac{\Pi_Z(0)}{M_Z^2} - \frac{\Pi_W(0)}{M_W^2} + 2\,\frac{\sin\Theta_{\rm W}}{\cos\Theta_{\rm W}}\,\frac{\Theta_{\gamma Z}(0)}{M_Z^2}$, and is dominated by the leading top-quark contribution, $\Delta\rho = \frac{3\sqrt{2} m_t^2\, G_\mu}{16\pi^2}$

The effective $\alpha_{\rm QED}$, from now on denoted as $\alpha(s)$, is a given for given energy scale $\sqrt{s}$ in terms of the photon vacuum polarization (VP) self-energy correction $\Delta\alpha$ in Eq. (2) as [1, 3]

$$\alpha(s) = \frac{\alpha}{1-\Delta\alpha(s)}\,; \qquad \Delta\alpha(s) = \Delta\alpha_{\rm lep}(s) + \Delta\alpha^{(5)}_{\rm had}(s) + \Delta\alpha_{\rm top}(s)\,. \quad (4)$$

The perturbative lepton and top-quark contributions are not critical. They are

$$\Delta\alpha_{\rm lep}\left(M_Z^2\right) \simeq 0.031419187418\,, \quad (5)$$
$$\Delta\alpha_{\rm top}\left(M_Z^2\right) = -0.76\times 10^{-4} \quad (6)$$

for the leptonic, and top-quark parts, respectively.

Interestingly, neglecting subleading electroweak corrections in Eq. (2) leads to about thirty standard deviations from the SM predictions and measured values of observables [3, 6].

In Tables 1 and 2, we summarized the status of higher-order corrections to the so-called electroweak pseudo-observables (EWPO) relevant for future collider studies, in particular FCC-ee. Intrinsic error in Table 1 indicates estimates for theoretical errors due to missing higher-order corrections. In Table 2, parametric errors are due to uncertainties in input parameters.

Table 1. Estimated precision for directly determining several important EWPO at FCC-ee. The last column shows the estimated projected intrinsic theory errors when leading 3-loop corrections become available. Presently, complete two-loop SM corrections are known [7, 8].

Quantity	FCC-ee	Current intrinsic error		Projected intrinsic error (at start of FCC-ee)
M_W [MeV]	0.5–1 ‡	4	$(\alpha^3, \alpha^2\alpha_{\rm s})$	1
$\sin^2\theta^{\ell}_{\rm eff}$ [10^{-5}]	0.6	4.5	$(\alpha^3, \alpha^2\alpha_{\rm s})$	1.5
Γ_Z [MeV]	0.1	0.4	$(\alpha^3, \alpha^2\alpha_{\rm s}, \alpha\alpha_{\rm s}^2)$	0.15
R_b [10^{-5}]	6	11	$(\alpha^3, \alpha^2\alpha_{\rm s})$	5
R_l [10^{-3}]	1	6	$(\alpha^3, \alpha^2\alpha_{\rm s})$	1.5

As we can see in Table 2, $\Delta\alpha$ is relevant so we will focus on the main term $\Delta\alpha^{(5)}_{\rm had}(s)$ in Eq. (4). For a more complete discussion of theoretical errors, see [8] and [9]. The non-perturbative hadronic VP shift $\Delta\alpha^{(5)}_{\rm had}(s)$ from the five light quarks and the hadrons they build is the main issue which influences not only precision high-energy collider physics but also low-energy predictions, in particular anomalous magnetic moments of leptons.

Table 2. Relevance of parametric uncertainty of several important EWPO due to uncertainties of input parameters m_t, m_b, M_Z, and $\Delta\alpha, \alpha_s$ which appear to be the main source of inaccuracies, as indicated in the fourth column.

Quantity	FCC-ee	Future parametric unc.	Main source
M_W [MeV]	0.5–1	1 (0.6)	$\delta(\Delta\alpha)$
$\sin^2\theta^{\ell}_{\rm eff}$ [10^{-5}]	0.6	2 (1)	$\delta(\Delta\alpha)$
Γ_Z [MeV]	0.1	0.1 (0.06)	$\delta\alpha_{\rm s}$
R_b [10^{-5}]	6	< 1	$\delta\alpha_{\rm s}$
R_ℓ [10^{-3}]	1	1.3 (0.7)	$\delta\alpha_{\rm s}$

The non-perturbative hadronic piece from the five light quarks $\Delta\alpha^{(5)}_{\rm had}(s) = -\left(\Pi'_\gamma(s) - \Pi'_\gamma(0)\right)^{(5)}_{\rm had}$ can be evaluated in terms of $\sigma(e^+e^- \to \text{hadrons})$ data via the dispersion integral (s *can be any, also negative!)*

$$\Delta\alpha^{(5)}_{\rm had}(s) \;=\; -\frac{\alpha\, s}{3\pi}\left(\int\limits_{m^2_{\pi^0}}^{E^2_{\rm cut}} {\rm d}s'\,\frac{R^{\rm data}_\gamma(s')}{s'(s'-s)} + \int\limits_{E^2_{\rm cut}}^{\infty} {\rm d}s'\,\frac{R^{\rm pQCD}_\gamma(s')}{s'(s'-s)}\right), \qquad (7)$$

where $R_\gamma(s) = \sigma^{(0)}(e^+e^- \to \gamma^* \to \text{hadrons})/\left(\frac{4\pi\alpha^2}{3s}\right)$ is the normalized hadronic cross section related to the tree-level $e^+e^- \to \mu^+\mu^-$ cross section in the high-energy limit $s \gg 4m^2_\mu$. Parametrization of $R_\gamma(s)$ is complicated, see for instance, [10, 11] where details can be found on the numerical evaluation of irreducible two-loop QED corrections for Bhabha scattering which include $R_\gamma(s)$. In practice, $R_\gamma(s)$ is divided into the perturbative and low-energy non-perturbative parts, including narrow resonances. It is worth noting that to implement narrow resonances properly, the VEGAS routine has been used in [12]. To improve the evaluation of $\Delta\alpha^{(5)}_{\rm had}(s)$ in Eq. (7), the Euclidean trick is used in the non-perturbative region, integrating dispersion relation for $\Delta\alpha^{(5)}_{\rm had}(s)$ with low-energy data at some negative energy $s = -M_0^2$. Moreover, defining auxiliary, a so-called Adler function, as the derivative of the VP function: $R(s) \longrightarrow D(-s) \equiv \frac{3\pi}{\alpha}\, s\, \frac{\rm d}{{\rm d}s}\Delta\alpha_{\rm had}(s) = -\left(12\pi^2\right)\, s\, \frac{{\rm d}\Pi'_\gamma(s)}{{\rm d}s}$, $D(s)$ can be also evaluated in terms of e^+e^--annihilation data by the dispersion integral

$$D\left(Q^2\right) \;=\; Q^2\left(\int\limits_{4m^2_\pi}^{E^2_{\rm cut}} {\rm d}s\,\frac{R(s)^{\rm data}}{(s+Q^2)^2} + \int\limits_{E^2_{\rm cut}}^{\infty} {\rm d}s\,\frac{R^{\rm pQCD}(s)}{(s+Q^2)^2}\right). \qquad (8)$$

$D(Q^2)$ is simpler to evaluate than $\Delta\alpha_{\rm had}$ in initial Eq. (7) and the profile of the dispersion integral is similar to that defined for $a_\mu^{\rm had}$

$$a_\mu^{\rm had} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int\limits_{4m_\pi^2}^{\infty} {\rm d}s\, \frac{R(s)\,\hat{K}(s)}{s^2}\,, \qquad \hat{K}(s) \in 0.63 \div 1\,. \tag{9}$$

Thus, improving $a_\mu^{\rm had}$ automatically leads to an improvement of $\Delta\alpha_{\rm had}^{(5)}$, where $\sigma(e^+e^- \to \pi^+\pi^-)$ cross section contributes more than 50% of the total hadronic VP to a_μ. There is persisting tension among main low-energy experiments concerning pion production processes. From the comparative analysis given in [13] follows that the biggest difference is between KLOE and BaBar measurements, which amounts to about 2%. It goes even up to 10% around the narrow ω resonance. For higher $\pi^+\pi^-$-invariant masses (at 0.9 GeV) the difference raises to 5%.

We can ask how far the inclusion of missed so far SM radiative corrections with dedicated kinematic cuts used in different experiments could cure the situation. Radiative corrections to the pion pair production have been considered in [14] and in [15] where measuring the FSR inclusive $\pi^+\pi^-$ cross section was analysed. In [16], the scalar QED (sQED) model has been considered, and pion form factors with FSR at NLO and pentaboxes were tested and implemented to the PHOKHARA 10.0 Monte Carlo generator [17]. Part of the results given in [16] with additional NLO contributions are shown in Fig. 1 for the KLOE case.

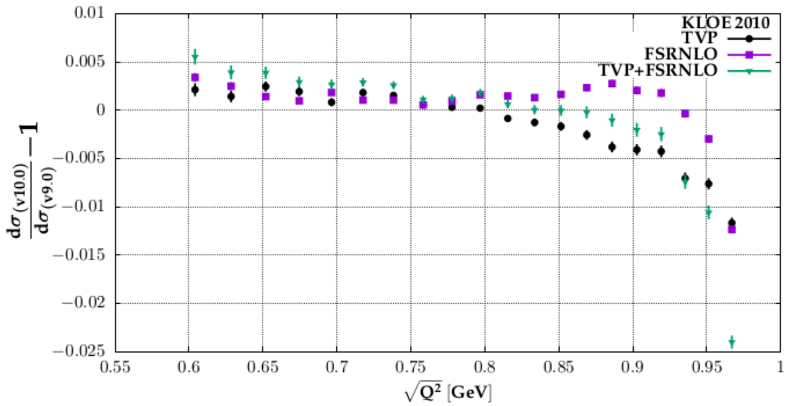

Fig. 1. The size of two-virtual-photon (TVP) and FSRNLO radiative corrections for KLOE 2010 with tagged photon event selections as a function of the $\pi^+\pi^-$-invariant mass Q^2. Figure taken from the `arXiv` version of [16].

In [16], it has been concluded that missing NLO radiative corrections cannot be the source of the discrepancies between the different extractions of the pion form factor performed by BaBar, BES, and KLOE. Consequently, they cannot be the origin of the discrepancy between the experimental measurement and the SM prediction of a_μ. We should mention that the BaBar Collaboration examined recently the measurement of additional radiation in the initial-state-radiation processes $e^+e^- \to \mu^+\mu^-\gamma$ and $e^+e^- \to \pi^+\pi^-\gamma$ [18]. They claim to differ from the results based on the PHOKHARA MC generator. However, the event selections used in [18] require at least two hard photons in the final state while the matrix elements in PHOKHARA for $e^+e^- \to \pi^+\pi^-\gamma\gamma$ and $e^+e^- \to \mu^+\mu^-\gamma\gamma$ are calculated at LO, so no surprise the accuracy is not high. For additional remarks on the subject, see [19].

The situation is dynamic concerning prospects for pion form factors determination. Recently, CMD-3 measurements [20] led to the reduction of tension between the experimental value of $a_\mu^{\rm had}$ and its Standard Model prediction, see further discussion on recent developments in a_μ and pion tensions in [21]. Let us also add that there is a new MUonE project at CERN where $\Delta\alpha_{\rm had}(-Q^2)$ and $\alpha(-Q^2)$ can be determined via μ^-e^- scattering [3] (the single number at $Q \sim 2.5$ GeV is expected). There is progress in the calculation of NNLO corrections which are mandatory for the success of this project [22]. Finally, precision of $\alpha(s)$ determination can be improved at the FCC-ee using $e^+e^- \to \mu^+\mu^-$ and forward–backward asymmetry, $A_{\rm FB}^{\mu\mu}$ [23]. The best accuracy is obtained for one year of FCC-ee running, below or above the Z pole, at 87.9 and 94.3 GeV, respectively.

3. Testing LFV processes at high and low energies

It has been argued long before LHC was built that heavy neutrinos (HN) can be probed at hadron colliders [24]. In Fig. 2, such a process $pp \to lljj$ is shown, accompanied by two other processes (in internal frame), $ll \to WW$ (future colliders) and $W^-W^- \to e^-e^-$ (part of the low-energy neutrinoless nuclear double beta decay $(\beta\beta)_{0\nu}$). For Majorana neutrinos, production of the same-sign leptons $pp \to W_2^\pm \to l_i^\pm N_a \to l_i^\pm l_j^\pm jj$ is possible (so LFV signals), while the other two processes are genuine LFV processes. All of them give a chance for detailed analysis of heavy Majorana neutrinos and relations to Dirac neutrinos for masses up to $M_N \sim \mathcal{O}(1)$ TeV [25, 26]. Heavy neutrinos are crucial for $(\beta\beta)_{0\nu}$ and colliders studies, and they are also naturally needed for standard leptogenesis [27]. Interestingly, the last studies show that heavy Majorana neutrinos can be probed at the LHC up to 10–20 TeV [28], see Fig. 3.

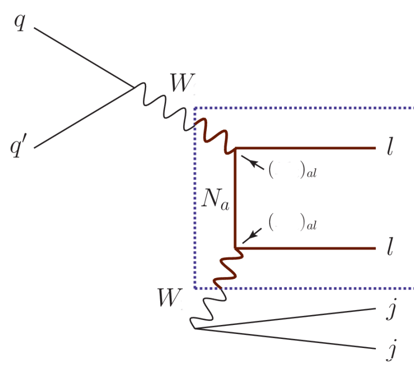

Fig. 2. This picture shows three processes which connect signals at high-energy colliders with low-energy LFV $(\beta\beta)_{0\nu}$. Possible light–heavy neutrino mixings are denoted as $(\ )_{al}$. See the main text for details.

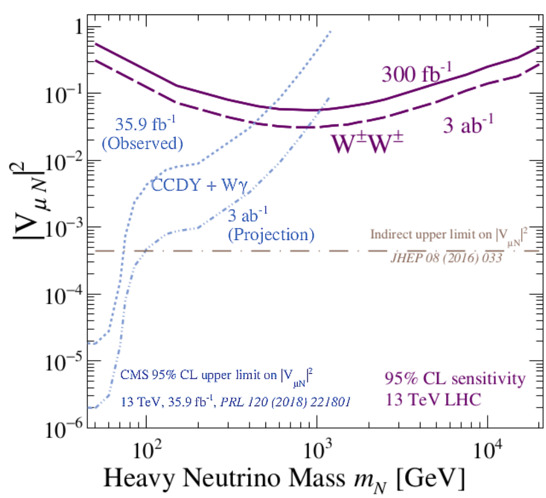

Fig. 3. Prospects for studying heavy neutrinos and LFV $W^\pm W^\pm$ signals at the LHC [28]. The figure taken from the `arXiv` version of [28].

Predictions for signals involving HN N_a depend on their mixing with light neutrinos, which transfer to the mixing with charged leptons l, V_{al}. Typically, global fits give $\kappa^2 = \sum_{a=\text{heavy}} V_{ae}^2 \leq 0.0054$ [29]. In addition, neutrino mixings depend on CP parities of heavy neutrinos [30]. For CP-conserving cases, we can relate CP parities of HN directly with light–heavy (LH) mixings, *e.g.* for the case of three HN, $M_{N_1} = M,\ M_{N_2} = AM,\ M_{N_3} = BM$, and $\eta_{\mathrm{CP}}(N_1) = \eta_{\mathrm{CP}}(N_2) = -\eta_{\mathrm{CP}}(N_3) = +i$, it follows that $V_{eN_1} \equiv x_1,\ V_{eN_2} \equiv x_2$

(both real), and $V_{eN_3} \equiv ix_3$ (complex). In this case, single LH mixing in the limit $M \leq 1$ TeV is [31]

$$| V_{Ne} |^2_{\max} = \frac{\kappa^2}{2} \,. \tag{10}$$

The largest mixing is possible for almost degenerate heavy neutrinos with not the same CP parities (to avoid $(\beta\beta)_{0\nu}$ Majorana constraint), with $A \to 1$ (two HN case), or $A \gg B, B \to 1$ (three HN case) [31].

In Fig. 4, we show an optimized signal for the LFV $e^-e^- \to W^-W^-$ process.

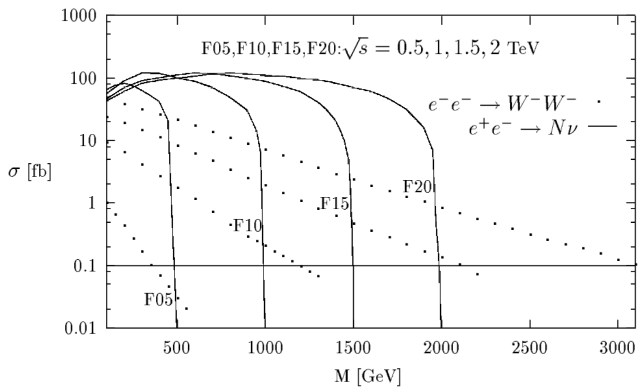

Fig. 4. Possible heavy-neutrino signals at e^+e^- [32] and e^-e^- colliders [31] as a function of the lightest of heavy-neutrino mass M, three heavy neutrinos are considered. The cross sections for the $e^-e^- \to W^-W^-$ process are chosen to be the largest. The solid line parallel to the M axis gives the predicted 'detection limit' ($\sigma = 0.1$ fb) for both processes. The figure taken from the `arXiv` version of [31].

At the limit $m_{\mathrm{heavy}(a)} \gg \sqrt{s} \gg M_W$, we get [33]

$$\sigma(e^-e^- \to W^-W^-) = \frac{G_\mu^2 s^2}{4\pi} \left| \sum_{\nu(a)} (V_{ae})^2 \frac{m_a}{s} + \sum_{N(a)} (V_{ae})^2 \frac{1}{m_a} \right|^2 . \tag{11}$$

The expression with sums in Eq. (11) is the same as for the weak part of the $(\beta\beta)_{0\nu}$ amplitude (so a constraint from the low energy process follows).

For the $e^+e^- \to N\nu$ reaction, the cross section in Fig. 4 is calculated for each HN mass using the same parameters as for $\sigma(e^-e^- \to W^-W^-)$. Such correlated cross sections for single HN production at e^+e^- colliders are not the biggest. For maximal single HN production, see [32]. Recently, interest has been revived in such processes for relatively small HN masses (below M_Z) in the context of Dirac and Majorana neutrino studies [34].

We should mention that the e^-e^- option is not considered seriously as a possible future collider mode. Interestingly, as shown in [31], there are regions of HN masses for which observable $e^+e^- \to N\nu$ signals are not possible but the $\Delta L = 2$ process $e^-e^- \to W^-W^-$ is still possible. It is a small region of 1 TeV $< M <$ 1.1 TeV for $\sqrt{s} = 1$ TeV, 1.5 TeV $< M <$ 2 TeV for $\sqrt{s} = 1.5$ TeV and 2 TeV $< M <$ 3.1 TeV for $\sqrt{s} = 2$ TeV where the cross section $\sigma\left(e^-e^- \to W^-W^-\right)$ is still above the background.

4. Conclusions

We discussed the need for improvements in determining input SM parameters as required by future high-energy e^+e^- colliders. We focused on the α_{QED} parameter and $\Delta\alpha_{\mathrm{had}}$ for which precise determination of the low-energy pion pair production is crucial. We mentioned existing tensions between experiments, particularly KLOE, BaBar, and CMD-3. We argued that calculated SM quantum corrections are insufficient to influence results and explain the a_μ discrepancy between experimental results and theory. Next, we discussed LFV hadron $pp \to lljj$ and lepton $e^-e^- \to W^-W^-$ collider high-energy processes, which involve heavy neutrinos and showed their connection with low-energy LFV $(\beta\beta)_{0\nu}$ process. We stressed the importance of considering various CP parities of heavy neutrinos, which affect predictions for light–heavy neutrino mixings, and discussed high- and low-energy processes.

The research has been supported by the National Science Center (NCN), Poland under grant 2023/50/A/ST2/00224.

REFERENCES

[1] R.L. Workman *et al.*, *Prog. Theor. Exp. Phys.* **2022**, 083C01 (2022).
[2] A. Abada *et al.*, *Eur. Phys. J. Spec. Top.* **228**, 261 (2019).
[3] F. Jegerlehner, *CERN Yellow Reports: Monographs* **3**, 9 (2020).
[4] M. Dong *et al.*, `arXiv:1811.10545 [hep-ex]`.
[5] P. Bambade *et al.*, `arXiv:1903.01629 [hep-ex]`.
[6] J.D. Wells, `arXiv:hep-ph/0512342`.
[7] I. Dubovyk *et al.*, *Phys. Lett. B* **783**, 86 (2018), `arXiv:1804.10236 [hep-ph]`.
[8] I. Dubovyk *et al.*, *J. High Energy Phys.* **2019**, 113 (2019), `arXiv:1906.08815 [hep-ph]`.
[9] A. Freitas *et al.*, `arXiv:1906.05379 [hep-ph]`.
[10] S. Actis, M. Czakon, J. Gluza, T. Riemann, *Phys. Rev. Lett.* **100**, 131602 (2008), `arXiv:0711.3847 [hep-ph]`.
[11] S. Actis, M. Czakon, J. Gluza, T. Riemann, *Phys. Rev. D* **78**, 085019 (2008), `arXiv:0807.4691 [hep-ph]`.

[12] C. Carloni Calame *et al.*, *J. High Energy Phys.* **2011**, 126 (2011), arXiv:1106.3178 [hep-ph].
[13] KLOE-2 Collaboration (A. Anastasi *et al.*), *J. High Energy Phys.* **2018**, 173 (2018), arXiv:1711.03085 [hep-ex].
[14] A. Hoefer, J. Gluza, F. Jegerlehner, *Eur. Phys. J. C* **24**, 51 (2002), arXiv:hep-ph/0107154.
[15] J. Gluza, A. Hoefer, S. Jadach, F. Jegerlehner, *Eur. Phys. J. C* **28**, 261 (2003), arXiv:hep-ph/0212386/.
[16] F. Campanario *et al.*, *Phys. Rev. D* **100**, 076004 (2019), arXiv:1903.10197 [hep-ph].
[17] PHOKHARA, a Monte Carlo event generator, https://looptreeduality.csic.es/phokhara/
[18] BABAR Collaboration (J.P. Lees *et al.*), *Phys. Rev. D* **108**, L111103 (2023), arXiv:2308.05233 [hep-ex].
[19] «PHOKHARA», talk by H. Czyż, 2nd Liverpool Workshop on Muon Precision Physics 2023 (MPP2023), https://indico.ph.liv.ac.uk/event/1297/
[20] CMD-3 Collaboration (F.V. Ignatov *et al.*), *Phys. Rev. D* **109**, 112002 (2024), arXiv:2302.08834 [hep-ex].
[21] G. Colangelo, M. Hoferichter, P. Stoffer, *PoS* **Muon4Future2023**, 019 (2024).
[22] A. Broggio *et al.*, *J. High Energy Phys.* **2023**, 112 (2023), arXiv:2212.06481 [hep-ph].
[23] P. Janot, *J. High Energy Phys.* **2016**, 053 (2016), arXiv:1512.05544 [hep-ph]; *Erratum ibid.* **2017**, 164 (2017).
[24] W.-Y. Keung, G. Senjanović, *Phys. Rev. Lett.* **50**, 1427 (1983).
[25] J. Gluza, *Acta Phys. Pol. B* **33**, 1735 (2002), arXiv:hep-ph/0201002.
[26] J. Gluza, T. Jeliński, R. Szafron, *Phys. Rev. D* **93**, 113017 (2016), arXiv:1604.01388 [hep-ph].
[27] G. Chauhan *et al.*, arXiv:2310.20681 [hep-ph].
[28] B. Fuks *et al.*, *Phys. Rev. D* **103**, 055005 (2021), arXiv:2011.02547 [hep-ph].
[29] F. del Aguila, J. de Blas, M. Pérez-Victoria, *Phys. Rev. D* **78**, 013010 (2008), arXiv:0803.4008 [hep-ph].
[30] S.M. Bilenky, S.T. Petcov, *Rev. Mod. Phys.* **59**, 671 (1987); *Erratum ibid.* **60**, 575 (1988); *Erratum ibid.* **61**, 169 (1989).
[31] J. Gluza, M. Zrałek, *Phys. Rev. D* **52**, 6238 (1995), arXiv:hep-ph/9502284.
[32] J. Gluza, J. Maalampi, M. Raidal, M. Zrałek, *Phys. Lett. B* **407**, 45 (1997), arXiv:hep-ph/9703215.
[33] J. Gluza, M. Zralek, *Phys. Lett. B* **362**, 148 (1995), arXiv:hep-ph/9507269.
[34] A. Blondel, A. de Gouvêa, B. Kayser, *Phys. Rev. D* **104**, 055027 (2021), arXiv:2105.06576 [hep-ph].