DOUBLE-HEAVY HADRONS IN THE BORN–OPPENHEIMER APPROXIMATION AND BEYOND*

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We briefly review the general procedure for building the Born–Oppenheimer Effective Field Theory Lagrangian at next-to-leading order and focus on the hyperfine splitting of heavy-quarkonium hybrids as an example. We use an interpolation between the known forms of the short- and long-distance potentials. We correct an error in one of the long-distance potentials, which leads to very small modifications of the previous results.

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1. Introduction

In quantum chromodynamics (QCD), delving into heavy-quarkonium systems has been a persistent endeavor from the outset of the theory. The significance of these heavy quarks, exceeding the QCD energy scale in mass, was promptly acknowledged. Within heavy hadrons, these particles move slowly, making them suitable for description using the standard non-relativistic quantum mechanics once the interaction potential is obtained from QCD, and this could be done in terms of the expectation value of the Wilson loop. Further refinement efforts have focused on accounting for corrections beyond leading order (LO), expressed through operator insertions in the Wilson loop. These pursuits culminated in the formulation of non-relativistic effective field theories (EFT), accommodating hard corrections [1] and providing a complete result for the potential up to $\mathcal{O}(1/m_Q^2)$ [2]. Notably, the diverse facets of the potential have been rigorously scrutinized through lattice QCD computations [3].

An analogous EFT for heavy quarkonium is detailed in [4]. It extends to any heavy hadron containing heavy quark–antiquark or two heavy quarks, alongside gluons and light quarks. These last two collectively termed

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light degrees of freedom (LDF) contribute to forming heavy exotic hadrons. A non-relativistic bound state is formed by the heavy quarks subject to an interaction potential depending on the LDF state. Three scales characterize these bound states: the heavy-quark mass m_Q, the relative momentum $m_Q v$, with relative velocity $v \ll 1$, and the binding energy $m_Q v^2$, whereas the LDF states are characterized by the typical hadronic scale $\Lambda_{\rm QCD}$. The EFT used is in a way similar to the strong-coupling regime of potential NRQCD (pNRQCD) as the energy gaps between the scales of the bound states go as follows: $m_Q \gg \Lambda_{\rm QCD}$ (NRQCD), $\Lambda_{\rm QCD} \gg m_Q v^2$, and $m_Q v \sim 1/r \sim \Lambda_{\rm QCD}$.

The LO of the adiabatic expansion between the dynamics of the heavy degrees of freedom, the heavy quarks, and the LDF due to $\Lambda_{\rm QCD} \gg m_Q v^2$ is the so-called Born–Oppenheimer approximation used in a non-relativistic EFT framework (BOEFT) in [5, 6]. The BOEFT has also been extended to include spin-dependent operators up to LO $1/m_Q$ [7, 8] and up to next-to-leading order (NLO) $1/m_Q^2$ [9].

We shall focus here on the calculation of the hyperfine splitting for hybrids as an example of an NLO calculation. A similar calculation for double-heavy baryons can be found in [10]. In order to compute the hyperfine splittings (HFS) for the lower-lying charmonium and bottomonium hybrids at LO in the BOEFT [8], we used an interpolation between the known form at short distances of the spin-dependent potentials given in [9] and the long-distance estimation using the QCD effective string theory [11]. We provide the results of the spectrum of the lower-lying static hybrid states (Σ_u and Π_u). We use the charmonium spectrum from [12] to determine the unspecified parameters in the short-distance form of the potentials and to evaluate the interpolation dependence. Thereafter, we are able to predict the HFS of higher multiplets and bottomonium hybrids.

2. The potentials

In Ref. [4], general expressions for the BOEFT at NLO were found and in [8], the heavy-quark spin-dependent terms are explicitly shown in the case of lower-lying hybrids, namely for $\kappa^p = 1^+$ as the quantum numbers of the LDF. We find that only two independent potentials describe the hyperfine splitting at LO, which can be arranged as follows:

$$V_{\rm hf}(r) = \frac{1}{6} V^{sa}_{1^+11}(r) - \frac{1}{3} V^{sb}_{1^+10}(r)\,, \qquad V_{\rm hf2}(r) = -\frac{1}{2}\left(V^{sa}_{1^+11}(r) + V^{sb}_{1^+10}(r)\right). \tag{1}$$

The potentials $V^{sa}_{1^+11}(r)$ and $V^{sb}_{1^+10}(r)$ at long distances are obtained using the QCD effective string theory (EST) [13], following the mapping given in Ref. [14]. We obtain

$$\frac{V^{sa}_{1^+11}(r)}{m_Q} = -\frac{2c_F\pi^2 g\Lambda'''}{m_Q\kappa r^3} \equiv V^{sa}_{\rm ld}(r)\,, \qquad \frac{V^{sb}_{1^+10}(r)}{m_Q} = \mp\frac{c_F g\Lambda' 2\pi^2}{m_Q\sqrt{\pi\kappa}}\frac{1}{r^2} \equiv V^{sb}_{\rm ld}(r)\,. \tag{2}$$

Parameters $g\Lambda' \sim -59$ MeV and $g\Lambda''' \sim \pm 230$ MeV as in [5]. $\kappa \simeq 0.187$ GeV^2 is the string tension and the short distance matching coefficient $c_F(m_c) \equiv c_F(\nu = 1\,\text{GeV}, m_c) = 1.12155$ and $c_F(m_b) \equiv c_F(\nu = 1\,\text{GeV}, m_b) = 0.87897$. We correct in (2) a factor of two in $V^{sb}_{1^+10}(r)$ overlooked in [8]. Due to the fact that $g\Lambda'$ is about four times smaller than $g\Lambda'''$, this correction has a very small impact on the results. In any case, we present the corrected spectrum here and comment on the differences with respect to [8].

The short-distance behavior has been studied in Refs. [9, 15]

$$V_{\rm hf}(r)/m_Q = A + \mathcal{O}\left(r^2\right)\,, \qquad V_{\rm hf2}(r)/m_Q = Br^2 + \mathcal{O}\left(r^4\right)\,. \tag{3}$$

$A = c_F\, k_A/m_Q$ and $B = c_F\, k_B/m_Q$ are the unknown real constants in the interpolation (4) that will be fit to lattice data. $k_A \sim \Lambda^2_{\rm QCD}$ and $k_B \sim \Lambda^4_{\rm QCD}$ as in [15].

Interpolating between the short- and the long-distance behavior with

$$\begin{aligned}
\frac{V_{\rm hf}(r)}{m_Q} &= \frac{A + \left(\frac{r}{r_0}\right)^2\left(\frac{1}{6}V^{sa}_{\rm ld}(r_0) - \frac{r}{3r_0}V^{sb}_{\rm ld}(r_0)\right)}{1+\left(\frac{r}{r_0}\right)^5}\,,\\
\frac{V_{\rm hf2}(r)}{m_Q} &= \frac{Br^2 - \left(\frac{r}{r_0}\right)^5\left(\frac{r_0}{2r}V^{sa}_{\rm ld}(r_0) + \frac{1}{2}V^{sb}_{\rm ld}(r_0)\right)}{1+\left(\frac{r}{r_0}\right)^7}\,,
\end{aligned} \tag{4}$$

where $r_0 \sim 1/\Lambda_{\rm QCD}$ is the matching scale. It is estimated from the short- and long-distance behavior of the static hybrid potentials to be $r_0 \simeq 3.96$ GeV^{-1}.

3. Charmonium hybrids spectrum

We fix the parameters A and B by comparing the results of the spectrum obtained with our interpolations (4) now with the correct long-distance behavior (2) to the lattice data of [12] for the lower-lying hybrid states.

Using $m_c = 1.47$ GeV, scanning values of A and B in the intervals $[-0.3\,, 0.3]$ GeV and $[-0.06\,, 0.06]$ GeV^3, respectively, and searching for the ones with the lowest χ^2/d.o.f., we obtain similar results as in [8] with no significant discrepancy. Using the same methodology as in [8], the best fit corresponds to a negative $V^{sa}_{\rm ld}(r)$ and a positive $V^{sb}_{\rm ld}(r)$. The former implies $g\Lambda''' < 0$. See Table 1.

Table 1. Fit parameters dependence on the sign ambiguities.

$\mathrm{sign}(V_{\mathrm{ld}}^{sa})$ $\mathrm{sign}(V_{\mathrm{ld}}^{sb})$	$+-$	$++$	$-+$	$--$
χ^2/d.o.f.	0.8796	0.7199	0.6291	0.7127
A [GeV]	0.0710	0.0925	0.1403	0.1205
B [GeV^3]	0.0107	0.0157	−0.00005	−0.0057

We explored the dependence of the result on $g\Lambda'$ and $g\Lambda'''$. The preference of the fit, is again, for larger values of $|g\Lambda'|$ and $|g\Lambda'''|$ (χ^2/d.o.f. = 0.619 *versus* χ^2/d.o.f. = 0.634). Therefore, we take for now on $g\Lambda' = -0.0796$ GeV and $g\Lambda''' = 0.3105$ GeV. These values lead to $A = 0.1509$ GeV and $B = -0.0015$ GeV^3. The interpolation dependence is estimated by moving $r_0 \in [3\,,5]$ GeV^{-1}. With the correct long-distance potentials, the χ^2/d.o.f. is the lowest for the default value $r_0 = 3.96$ GeV^{-1}, in contrast to [8], where the χ^2/d.o.f. marginally improved around $r_0 \sim 3.5$ GeV^{-1}. See Table 2. Regarding the errors of A and B due to the input data and the error due to higher orders in the $1/m_Q$ expansion, we obtain $A = 0.124 \pm 0.034$ GeV and $B = 0.004 \pm 0.016$ GeV^3. The spectrum is displayed in Table 3.

Table 2. Fit parameters dependence on r_0.

r_0 [GeV^{-1}]	3	3.5	3.96	5
χ^2/d.o.f.	0.7834	0.6328	0.6195	0.7349
A [GeV]	0.2575	0.1877	0.1509	0.1049
B [GeV^3]	−0.0016	−0.0045	−0.0015	−0.0001

4. Bottomonium hybrids spectrum

With A and B fixed, the corresponding parameters A' and B' to predict the bottomonium hyperfine splittings are also fixed by a relation on the masses of charm and bottom. Using $m_b = 4.88$ GeV and computing the spectrum for the central values of these parameters $A' = 0.029 \pm 0.008$ GeV and $B' = 0.001 \pm 0.004$ GeV^3, we obtain the results displayed in Table 4. If we compare this table with the one obtained in [8], we see that the change in the hyperfine splitting is marginal.

Table 3. Fit errors in A, B, and in the hybrid charmonium spectrum. The total errors in the spectrum are obtained by adding in quadrature the error due to missing higher orders in the $1/m_Q$ expansion ($\sim \Lambda_{\rm QCD}^3/m_Q^2 \sim 30$ MeV).

			Fit error	Total error
A [GeV]		0.124	0.034	
B [GeV^3]		0.004	0.016	
$(s/d)_1$ mass [GeV]	1^{--}	4.011		0.030
	0^{-+}	3.907	0.046	0.055
	1^{-+}	3.962	0.025	0.039
	2^{-+}	4.046	0.018	0.035
p_1 mass [GeV]	1^{++}	4.145		0.030
	0^{+-}	4.091	0.056	0.063
	1^{+-}	4.054	0.024	0.038
	2^{+-}	4.124	0.005	0.030
$(p/f)_2$ mass [GeV]	2^{++}	4.232		0.030
	1^{+-}	4.237	0.019	0.036
	2^{+-}	4.260	0.021	0.036
	3^{+-}	4.240	0.013	0.033
p_0 mass [GeV]	0^{++}	4.486		0.030
	1^{+-}	4.449	0.013	0.033

Table 4. The hybrid bottomonium spectrum. The total errors in the spectrum are obtained by adding in quadrature to the fit errors the error due to missing higher orders in the $1/m_Q$ expansion ($\sim \Lambda_{\rm QCD}^3/m_Q^2 \sim 3$ MeV).

		Mass [GeV]	A and B error	Total error
$(s/d)_1$	1^{--}	10.690		0.003
	0^{-+}	10.682	0.004	0.005
	1^{-+}	10.686	0.002	0.004
	2^{-+}	10.694	0.002	0.004
p_1	1^{++}	10.761		0.003
	0^{+-}	10.756	0.004	0.005
	1^{+-}	10.759	0.002	0.004
	2^{+-}	10.764	0.002	0.003
$(p/f)_2$	2^{++}	10.819		0.003
	1^{+-}	10.815	0.002	0.003
	2^{+-}	10.818	0.000	0.003
	3^{+-}	10.821	0.001	0.003
p_0	0^{++}	11.012		0.003
	1^{+-}	11.012	0.000	0.003

5. Conclusions

The benefits of interpolating between short- and long-distance potentials are outlined in Ref. [8]. In summary, significant improvement in describing the hyperfine splittings of charmonium hybrids from lattice QCD data is observed when long-distance contributions computed with QCD EST are included in LO spin-dependent potentials within BOEFT. Comparing with the fit using the NLO short-distance potentials [15], the χ^2/d.o.f. moves from 0.999 in that fit to 0.619 in ours. With the unknown parameters fixed, we computed the hyperfine splittings of higher charmonium hybrid states, of the bottomonium ones, and the error associated with them.

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