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STATIC HYBRID POTENTIALS FROM LAPLACIAN EIGENMODES*

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We test a method for computing static hybrid quark–antiquark potentials in lattice QCD, which is not based on Wilson loops, but where the trial states are formed by eigenvector components of the covariant lattice Laplace operator and their covariant derivatives. The main advantage of the new method is that we can compute off-axis distances without much extra work and by introducing a basis of Gaussian profiles in distillation space and solving a GEVP, we can also access excited states. We present a static hybrid spectrum with ground and excited Σ and Π states, and perform a basic Born–Oppenheimer approximation on a QCD-like ensemble with two dynamical quarks of half the charm quark mass.

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1. Introduction

A system comprising an infinitely heavy quark and antiquark pair immersed in the QCD vacuum with light quarks has very rich physics, describing both the dynamics of confinement and the decay of heavy-quarkonium states into heavy-flavoured meson pairs [1]. The energy of this system varies as the separation between the static colour sources is varied and for short distances, this energy is computed in lattice QCD using operators describing a flux tube connecting the sources. The simplest such operator is a straight Wilson line [2] joining the two sources and this creates the ground-state gluon flux tube [3]. In [4, 5], we have introduced new operators, namely "Laplace trial states", which replace the spatial Wilson line with a weighted sum of

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eigenvector pairs of the 3D lattice Laplace operator. Here, we want to apply this technique to compute static-hybrid potentials. The latter represent a promising frontier in the quest to decipher the strong force's intricacies. These potentials offer a unique perspective on meson behavior, bridging the gap between traditional quark models and lattice QCD simulations [6]. In particular, XYZ states have posed significant challenges to our understanding of hadron spectroscopy. These states do not fit neatly into the quark model, and their properties pose questions about their internal structure and how they relate to QCD [7]. Static-hybrid mesons could provide a theoretical framework to help explain or categorize some XYZ states. One hypothesis is that some XYZ states could be interpreted as hybrid mesons where gluonic excitations play a significant role. The "gluexcitement" of the strong force within these states may contribute to their exotic properties. The excited flux tube is adding non-zero angular momentum about the axis of the quark–antiquark pair. The quantum numbers of this system are the angular momentum around the inter-source axis, the discrete transformations combining charge conjugation with parity reflection about the central point of the system, and a spatial reflection with respect to a plane including the axis of separation. We show how to construct new static hybrid operators with trial states formed by eigenvector components of the covariant lattice Laplace operator, where the gluonic excitations are realized via covariant derivatives of individual eigenvectors.

2. Static hybrid potentials

The static hybrid potentials are characterized by the following quantum numbers Λ_η^ϵ [8]:

— $\Lambda = 0, 1, 2, 3, \ldots \equiv \Sigma, \Pi, \Delta, \Phi, \ldots,$ the absolute value of the total angular momentum with respect to the axis of separation of the static quark–antiquark pair,

— $\eta = +, - \equiv g, u$, the eigenvalue corresponding to the operator $\mathcal{P} \circ \mathcal{C}$, *i.e.* the combination of parity about the central point and charge conjugation,

— $\epsilon = +, -$, the eigenvalue corresponding to the operator $\mathcal{P}_x$, which denotes the spatial reflection with respect to a plane including the axis of separation.

Note that for angular momentum $\Lambda > 0$, the spectrum is degenerate with respect to ϵ. These quantum numbers are derived from the continuous group $D_{\infty h}$, which leaves a cylinder along a chosen axis invariant. The irreducible representations (irreps) of this group are $A_1^\pm$ (also denoted as $\Sigma_\pm^+$), $A_2^\pm$ ($\Sigma_\pm^-$),

$E_1^{\pm}\,(\Pi_{\pm})$, $E_2^{\pm}\,(\Delta_{\pm})$, $E_3^{\pm}\,(\Phi_{\pm})$, *etc.* On the lattice, we have only D_{4h} with 10 irreps: $A_1^{\pm}$, $A_2^{\pm}$, $B_1^{\pm}$, $B_2^{\pm}$ (all $d=1$), and $E^{\pm}$ ($d=2$). The subduction relations are given by $A_{1,2}^{\pm} \to A_{1,2}^{\pm}$, $E_1^{\pm} \to E^{\pm}$, $E_2^{\pm} \to B_1^{\pm} \oplus B_2^{\pm}$, *etc.*

In order to build hybrid Laplace trial states, we introduce gluonic excitations via covariant derivatives of the Laplacian eigenvectors

$$\nabla_{\vec{k}} v(\vec{x}\,) = \frac{1}{2}\left[U_k(\vec{x}\,)v\left(\vec{x}+\hat{k}\right) - U_k^{\dagger}\left(\vec{x}-\hat{k}\right)v\left(\vec{x}-\hat{k}\right)\right].$$

Derivative-based operators transform according to the 10 irreps of the cubic group O_h, *e.g.* $\nabla_i(T_1^-)$, $\mathbb{B}_i = \epsilon_{ijk}\nabla_j\nabla_k(T_1^+)$, $\mathbb{D}_i = |\epsilon_{ijk}|\nabla_j\nabla_k(T_2^+)$, $\mathbb{E}_i = \mathbb{Q}_{ijk}\nabla_j\nabla_k(E^+)$, $\nabla^2(A_1^+)$ [9]. We restrict ourselves to one derivative operator here. Since D_{4h} is a subgroup of O_h, we have the subduction relations $A_1^{\pm} \to A_1^{\pm}$, $A_2^{\pm} \to B_1^{\pm}$, $E^{\pm} \to A_1^{\pm} \oplus B_1^{\pm}$, $T_1^{\pm} \to A_2^{\pm} \oplus E^{\pm}$, and $T_2^{\pm} \to B_2^{\pm} \oplus E^{\pm}$. The fourth relation shows that the three components of ∇_i get separated into one that transforms like A_2 (along the separation axis) and two that transform like E (the two orthogonal to the separation axis). The gluonic excitations allow us to also access exotic quantum numbers, *i.e.* $J^{PC} = 0^{--}, 0^{+-}, 1^{-+}, 2^{+-}, 3^{-+}, 4^{+-}, 5^{-+}, \ldots$, which are not allowed in the pure quark model, where $P = (-1)^{L+1}$ and $C = (-1)^{L+S}$, with orbital angular momentum $L \in \{0,1,2,\ldots\}$ and spin $S \in \{0,1\}$. In this paper, we consider the following trial states with static quarks at $\vec{x}$ and $\vec{y}$, respectively, *i.e.*, a quark separation $R = |\vec{r}\,| = |\vec{y}-\vec{x}\,|$:

$$\Sigma_g^+(R) = v_i(\vec{y},t_0)v_i^{\dagger}(\vec{x},t_0)\,, \quad \text{or} \tag{1}$$

$$\Sigma_g^+(R) = \nabla_{\vec{k}||\vec{r}}v_i(\vec{y},t_0)v_i^{\dagger}(\vec{x},t_0) - v_i(\vec{y},t_0)\left[\nabla_{\vec{k}||\vec{r}}v_i\right]^{\dagger}(\vec{x},t_0)\,, \tag{2}$$

$$\Sigma_u^-(R) = \nabla_{\vec{k}||\vec{r}}v_i(\vec{y},t_0)v_i^{\dagger}(\vec{x},t_0) + v_i(\vec{y},t_0)\left[\nabla_{\vec{k}||\vec{r}}v_i\right]^{\dagger}(\vec{x},t_0)\,, \tag{3}$$

$$\Pi_g(R) = \nabla_{\vec{k}\perp\vec{r}}v_i(\vec{y},t_0)v_i^{\dagger}(\vec{x},t_0) - v_i(\vec{y},t_0)\left[\nabla_{\vec{k}\perp\vec{r}}v_i\right]^{\dagger}(\vec{x},t_0)\,, \tag{4}$$

$$\Pi_u(R) = \nabla_{\vec{k}\perp\vec{r}}v_i(\vec{y},t_0)v_i^{\dagger}(\vec{x},t_0) + v_i(\vec{y},t_0)\left[\nabla_{\vec{k}\perp\vec{r}}v_i\right]^{\dagger}(\vec{x},t_0)\,. \tag{5}$$

3. Results

We performed all our measurements on 48×24^3 lattices with periodic boundary conditions except for anti-periodic boundary conditions for the fermions in the temporal direction. They were produced with the openQCD package using the plaquette gauge action and two dynamical non-perturbatively $O(a)$ improved Wilson quarks with a mass equal to half of the physical charm quark mass (≈ 600 MeV). The bare gauge coupling is $g_0^2 = 6/5.3$ and the hopping parameter is $\kappa = 0.13270$. The lattice spacing is $a = 0.0658(10)$ fm. All measurements were performed by our C+MPI-based library that facilitates massively parallel QCD calculations. A total of $N_v = 200$ eigenvectors of the 3D covariant Laplacian were calculated on each time-slice of the lattices, as described in [10]. A total of 20 3D APE smearing steps with $\alpha_{\mathrm{APE}} = 0.5$ were applied on each gauge field before the eigenvector calculation so as to smooth the link variables that enter the Laplacian operator. When forming the correlations of the Laplace trial states, we apply one HYP2 smearing step with parameters $\alpha_1 = 1$, $\alpha_2 = 1$, and $\alpha_3 = 0.5$ to the temporal links [11], which corresponds to a particular choice of the static action.

We show the static hybrid ground-state potentials of $\Sigma^{\pm}_{g/u}$ and $\Pi_{g/u}$, and some excited states in figure 1. We plot the potentials relative to twice the static-light S-wave m_{ps} and also mark the P_{-}-wave mass m_s [12]. We see that expected string-breaking distances of Σ^{+}_{g} and Π_u are just above half the spatial lattice extent. For on-axis separations, the potential of $\Pi_{g/u}$ in the continuum representation can be obtained from the $E^{\mp}_{1}$ representation of D_{4h}. For off-axis separations, we technically do not have D_{4h}, but we can consider off-axis separations in a 2D plane only rather than the 3D volume, to be left with one orthogonal direction for the covariant derivatives. For Σ^{-}_{u} with derivatives along the separation axis, we compute on-axis distances only.

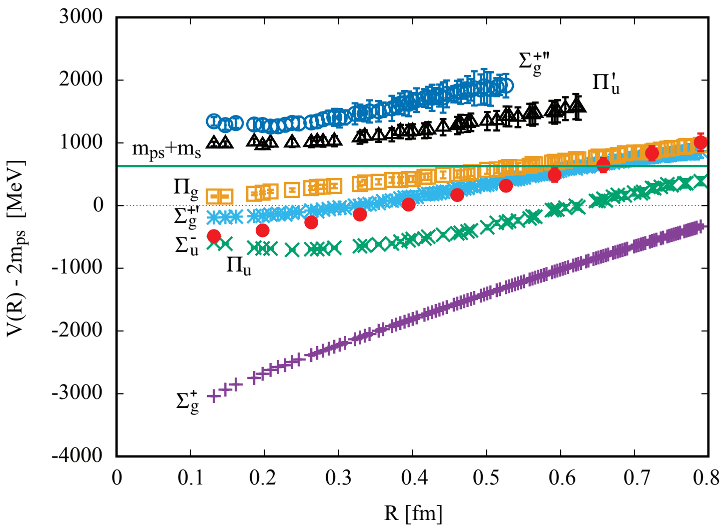

Fig. 1. Static hybrid potentials for $\Sigma^{\mp}_{u/g}$ and $\Pi_{u/g}$ relative to twice the static-light S-wave mass m_{ps} and the P_{-}-wave mass m_s, string breaking distances of Σ^{+}_{g} and Π_u are just above half the lattice extent. 3D resp. 2D off-axis distances for Σ^{+}_{g} and $\Pi_{g/u}$, on-axis only for Σ^{-}_{u}.

4. Born–Oppenheimer

For a simple proof of principle, the static (hybrid) potentials can be used in the Born–Oppenheimer approximation [13] to compute the spectrum of quarkonium by solving the non-relativistic Schrödinger equation in the adiabatic potential generated by the relativistic light quarks and intrinsic gluons

$$E_{\Lambda_\eta^\epsilon;L,n}\Psi_{\Lambda_\eta^\epsilon;L,n}(r) = \left(-\frac{1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{L(L+1) - 2\Lambda^2 + J_{\Lambda_\eta^\epsilon}(J_{\Lambda_\eta^\epsilon}+1)}{2\mu r^2} + V_{\Lambda_\eta^\epsilon}(r)\right)\Psi_{\Lambda_\eta^\epsilon;L,n}(r)\,, \quad (6)$$

where r is the separation of the heavy $\bar{Q}Q$ pair and $\mu = m_Q m_{\bar{Q}}/(m_Q + m_{\bar{Q}})$ its reduced quark mass. $L \in \{\Lambda, \Lambda+1, \ldots\}$ is the quantum number corresponding to the operator $\boldsymbol{L}$, the sum of all angular momenta excluding the heavy quark spins $\boldsymbol{S}$, *i.e.* $\boldsymbol{J} = \boldsymbol{L}+\boldsymbol{S}$, where $\boldsymbol{J}$ is the total angular momentum of the meson, *i.e.*, $J_{\Lambda_\eta^\epsilon} = 0$ for $\Lambda_\eta^\epsilon = \Sigma_g^+$, $J_{\Lambda_\eta^\epsilon} = 1$ for $\Lambda_\eta^\epsilon \in \{\Sigma_g^{+\prime}, \Pi_g, \Pi_u\}$ and $J_{\Lambda_\eta^\epsilon} = 2$ for $\Lambda_\eta^\epsilon = \Sigma_u^+$. In this adiabatic approximation, the gluon field is assumed to be in a stationary state in the presence of the heavy $\bar{Q}Q$ pair, with errors proportional to $\Lambda_{\mathrm{QCD}}/m_Q$, which is suited for heavy quarks. Further, only a single component of this multi-channel Schrödinger equation is considered and couplings to other channels are ignored, which is valid for small separations r, where the used hybrid static potential has avoided crossings with the other hybrid static potentials. We solve Eq. (6) numerically using $m_c = 1628$ MeV or $m_b = 4977$ MeV and for the potentials $V_{\Lambda_\eta^\epsilon}(r)$ the parametrizations suggested in [14]

$$V_{\Sigma_g^+}(r) = V_0 - \alpha/r + \sigma r \quad \text{and} \quad V_{\Pi_u}(r) = A_1/r + A_2 + A_3 r^2\,.$$

The energies $E_{\Lambda_\eta^\epsilon;L,n}$ contain the self-energies of the static quarks, which can be eliminated via

$$m_{\Lambda_\eta^\epsilon;L,n} = E_{\Lambda_\eta^\epsilon;L,n} - E_{\Sigma_g^+;L=0,n=1} + \bar{m}$$

with $\bar{m}$ the spin averaged mass $\bar{m}_c = (m_{\eta_c(1S),\mathrm{exp}} + 3m_{J/\Psi(1S),\mathrm{exp}})/4 = 3069(1)$ MeV and $\bar{m}_b = (m_{\eta_b(1S),\mathrm{exp}} + 3m_{\Upsilon(1S),\mathrm{exp}})/4 = 9445(1)$ MeV [15]. The energy $E_{\Lambda_\eta^\epsilon=\Sigma_g^+;n=1,L=0}$ corresponds for $\bar{Q}Q = \bar{c}c$ to the $\eta_c(1S)$ and $J/\Psi(1S)$ meson, which are degenerate in the static limit, and similarly for $\bar{Q}Q = \bar{b}b$ to the $\eta_b(1S)$ and $\Upsilon(1S)$ meson. In Table 1, we list bound-state energies of the static potential Σ_g^+ and the lowest Π_u state from the Born–Oppenheimer analysis together with their mean squared distance $\sqrt{\langle r^2\rangle}$ of the wave function ($\langle r^2\rangle = \int_0^\infty \Psi(r) r^2 \mathrm{d}r$). The former are compared with experimental values of the $m_{J/\Psi}(nS)$ and $m_{\Upsilon}(nS)$, the computed values are consistently larger in our non-physical setup; the latter, therefore, only provide a very vague prediction, also because $1/m_Q$ corrections, *e.g.* from the quark spins, are neglected.

Table 1. First two bound-state energies in MeV of the static potential Σ_g^+ and hybrid Π_u state from the Born–Oppenheimer analysis using $m_Q = m_c = 1628$ MeV (left) and $m_Q = m_b = 4977$ MeV (right). $E^1_{\Sigma_g^+}$ is given by $\bar{m}_{c/b}$, other Σ states are compared with experimental values of $m_{J/\Psi}(nS)$ and $m_{\Upsilon}(nS)$, respectively.

$m_Q = m_c$	$n = 1$	$n = 2$	$m_Q = m_b$	$n = 1$	$n = 2$
$L = 0,\ \Sigma_g^+$	$\bar{m}_c$	4347(19)	$L = 0,\ \Sigma_g^+$	$\bar{m}_b$	10402(17)
$m_{J/\Psi}(nS)$	3096.900(6)	3674(1)	$\Upsilon(nS)$	9460.4(1)	10023.4(5)
$L = 1,\ \Pi_u$	4504(12)		$L = 1,\ \Pi_u$	10992(12)	11531(13)

5. Conclusions and outlook

We have computed static hybrid potentials $V_{\Lambda_\eta^\epsilon}(r)$ for $\Lambda_\eta^\epsilon = \Sigma_{g/u}$ and $\Pi_{g/u}$ states in SU(3) lattice gauge theory using alternative operators for a static quark–antiquark pairs based on Laplacian eigenmodes, replacing traditional Wilson loops. Instead of "gluonic handles" (excitations) of the spatial Wilson lines, we use symmetric, covariant derivatives of the Laplacian eigenvectors to form improved Laplace trial states by applying optimal profiles to give different weights to individual eigenvectors, derived from a generalized eigenvector problem. A high resolution of the static hybrid potentials can be achieved as off-axis distances can easily be computed in the new approach. We present a static hybrid spectrum including excited states, where we also mark the string breaking masses from static-light S- and P-waves, the string breaking distances of Σ_g and Π_u are just above half of our lattice extent. The implementation of (hybrid) static-light correlators using "perambulators" $v(t_1)D^{-1}v(t_2)$ from the distillation framework, *i.e.*, quark field smearing via projection $\psi \to vv^\dagger\psi$ [10], is the subject of a forthcoming paper. This allows us to put together the building blocks for string breaking in QCD, computing the mixing matrix of static and light quark propagators [16]. The new methods can also be applied to tetra- and multi-quark potentials.

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