THE AXIAL CURRENT AND ITS DIVERGENCE*

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Received 29 November 2024, accepted 16 December 2024,
published online 6 March 2025

In the Standard Model, the axial current is not conserved due to fermion masses and the axial anomaly. In this work, we employ perturbative quantum chromodynamics to evaluate the matrix elements of the local and non-local axial currents for a gluon target, providing insights into their relation with the axial anomaly. Our analysis revisits well-established results related to the nucleon spin sum rule, along with recent developments in off-forward kinematics. A significant aspect of our approach is the use of an infrared regulator, with a particular focus on the non-zero quark mass. We observe important cancellations between the contributions from the axial anomaly and the quark mass term, and we discuss how these cancellations are linked to the conservation of angular momentum.

DOI:10.5506/APhysPolBSupp.18.1-A21

1. Introduction

The axial anomaly shows up in the divergence of the flavor-singlet axial current, $J_5^\mu(x) = \sum_q \bar{q}(x)\,\gamma^\mu\gamma_5\,q(x)$,

$$\partial_\mu J_5^\mu(x) = \sum_q 2im_q\,\bar{q}(x)\,\gamma_5\,q(x) - \frac{\alpha_{\rm s} N_f}{4\pi}\,{\rm Tr}\left(F^{\mu\nu}(x)\widetilde{F}_{\mu\nu}(x)\right)\,, \qquad (1)$$

which contains the quark mass term and the anomaly contribution. Soon after the discovery of the nucleon spin crisis, it was suggested that the axial

* Presented at the Diffraction and Low-x 2024 Workshop, Trabia, Palermo, Italy, 8–14 September, 2024.

anomaly could play an important role in understanding the nucleon's spin structure. Specifically, it was argued [1, 2] that the measured (very small) quark-spin contribution $\Delta\Sigma$ differs from the 'intrinsic' quark-spin contribution $\Delta\widetilde{\Sigma}$ according to

$$\Delta\Sigma = \Delta\widetilde{\Sigma} - \frac{\alpha_{\mathrm{s}} N_f}{2\pi}\,\Delta G\,, \tag{2}$$

where ΔG represents the gluon-spin contribution. The term proportional to ΔG was attributed to the axial anomaly. This proposal, however, faced several criticisms. For instance, it was argued [3] that the anomaly would give rise to a non-local contribution. In that regard, the concern was that matrix elements of the anomaly operator between gluon states would diverge in the collinear limit in which the two gluons have the same momentum. It was also suggested that this 'pole' behavior would provide an explanation for the generation of the large mass of the η' meson [4, 5].

Recent studies [6–9] have revisited this idea, discussing also deep-virtual Compton scattering (DVCS) off the proton in perturbative QCD. In this work, we adopt a similar approach but use the quark mass as an infrared (IR) regulator and perform explicit contractions with physical polarization vectors of the gluons. Contrary to earlier expectations, we do not observe problematic behavior in the forward limit. Furthermore, we find that the 'pole' term arising from the anomaly cancels with the contribution from the quark mass term. Interestingly, this cancellation in the collinear limit is actually required by the conservation of angular momentum. More details about our work can be found elsewhere [10].

2. Parton distribution function

We define the parton distribution function (PDF) $g_1(x)$ by evaluating the light-cone operator of the axial quark current between gluon states

$$\begin{aligned}\varPhi^{[\gamma^+\gamma_5]}_{\lambda\lambda'}(x) &= \int \frac{\mathrm{d}z^-}{4\pi}\mathrm{e}^{ik\cdot z}\left\langle g\left(p,\lambda'\right)\right|\bar{q}\left(-\tfrac{z}{2}\right)\gamma^+\gamma_5\mathcal{W}\left(-\tfrac{z}{2},\tfrac{z}{2}\right)q\left(\tfrac{z}{2}\right)\left|g(p,\lambda)\right\rangle\Big|_{\mathrm{lc}} \\ &= -\frac{i}{p^+}\varepsilon^{+\,\epsilon\,\epsilon'^*p}\,g_1(x)\,,\end{aligned} \tag{3}$$

with $\lambda\,(\lambda')$ denoting the polarization state of the incoming (outgoing) gluon, $\varepsilon^{+\,\epsilon\,\epsilon'^*p} = \varepsilon^{+\,\mu\,\nu\,\rho}\,\epsilon_\mu\,\epsilon'^*_\nu\,p_\rho$, and $x = k^+/p^+$. The PDF is given by

$$g_1(x) = \frac{1}{2}\left(\varPhi^{[\gamma^+\gamma_5]}_{++}(x) - \varPhi^{[\gamma^+\gamma_5]}_{--}(x)\right), \tag{4}$$

which also indicates that, as a consequence of angular momentum conservation, the matrix element in Eq. (3) is non-zero only for $\lambda = \lambda'$.

We compute $g_1(x)$ at $\mathcal{O}(\alpha_s)$ in perturbative QCD for a finite quark mass and space-like off-shellness p^2 of the gluons, finding for the positive x-region

$$g_1\left(x;m,p^2\right) = \frac{\alpha_s}{4\pi}\left[\left(\frac{1}{\varepsilon} - \ln\frac{m^2 - p^2x(1-x)}{\bar{\mu}^2}\right)(2x-1) + \frac{p^2x(1-x)}{m^2-p^2x(1-x)}\right], \tag{5}$$

where $\bar{\mu}^2 = 4\pi e^{-\gamma_E}\mu^2$ and μ is the regularization scale. The result for the negative x-region follows by substituting $x \to -x$. The UV divergence of g_1 is reflected by the $1/\varepsilon$ pole, while both m and p^2 act as IR regulators.

The lowest moment of $g_1(x)$, which gives the local axial current for forward kinematics, is UV-finite, but it depends on the $\eta = -p^2/m^2$ ratio [2]. Here, we just report results in two limits

$$\int_{-1}^{1} dx\, g_1\left(x;m,p^2\right) \overset{\eta\to 0}{\to} 0\,, \qquad \int_{-1}^{1} dx\, g_1\left(x;m,p^2\right) \overset{\eta\to\infty}{\to} -\frac{\alpha_s}{2\pi}\,. \tag{6}$$

Multiplying the expression in Eq. (6) for $\eta \to \infty$ by the number of quark flavors N_f gives the prefactor of ΔG in Eq. (2) [2]. In the next section, we connect the results in Eq. (6) to the operators on the r.h.s. of Eq. (1).

3. Local axial current

We move on to discuss the matrix element of the local axial current $J_5^\mu(x)$ for non-zero momentum transfer. To this end, we first consider the matrix element of the divergence of the current

$$\langle g\left(p,\lambda'\right)|\,\partial_\mu J_5^\mu(0)\,|g(p,\lambda)\rangle = -2\,\varepsilon^{\epsilon\,\epsilon'^*P\,\Delta}\left(D_a\left(\Delta^2\right) + D_m\left(\Delta^2\right)\right)\,, \tag{7}$$

with $P = (p+p')/2$ and $\Delta = p' - p$. The quantity D_a (D_m) is the contribution of the anomaly (mass) term in Eq. (1). We evaluate the matrix element in Eq. (7) for two cases: *(i)* arbitrary Δ^2, $m \neq 0$, on-shell gluons; *(ii)* zero Δ^2, $m \neq 0$, off-shell gluons ($p^2 = p'^2 < 0$). We obtain

$$\begin{aligned} D_a\left(\Delta^2;m,0\right) &= -\frac{\alpha_s}{2\pi}\,, \qquad D_m\left(\Delta^2;m,0\right) = \frac{\alpha_s}{2\pi}\,\frac{1}{\tau}\ln^2\frac{\sqrt{\tau+4}+\sqrt{\tau}}{\sqrt{\tau+4}-\sqrt{\tau}}\,, \\ D_m\left(0;m,p^2\right) &= -\frac{\alpha_s}{2\pi}\,, \\ D_m\left(0;m,p^2\right) &= \frac{\alpha_s}{2\pi}\,\frac{2}{\sqrt{\eta\,(\eta+4)}}\ln\frac{\sqrt{\eta+4}+\sqrt{\eta}}{\sqrt{\eta+4}-\sqrt{\eta}}\,, \end{aligned} \tag{8}$$

with $\tau = -\Delta^2/m^2$. For both $\tau \to 0$ and $\eta \to 0$, there is an exact cancellation between D_a and D_m [10]. We also find that indeed the axial anomaly contributes to both results in Eq. (6), where in the case of $\eta \to \infty$ the quark mass term is absent and the non-zero result first reported in Ref. [2] is entirely due to the axial anomaly.

The local axial current for on-shell (real) and off-shell (virtual) gluons takes the general form (see Ref. [10] and references therein)

$$\Gamma_5^\mu\big|_{\text{real}} = \left(G_1\left(\Delta^2; m, 0\right) + G_2\left(\Delta^2; m, 0\right)\right) A_2^\mu = G\left(\Delta^2; m, 0\right) A_2^\mu\,, \tag{9}$$

$$\Gamma_5^\mu\big|_{\text{virtual}} = -\frac{4p^2 G_1\left(\Delta^2; m, p^2\right)}{\Delta^2 - 4p^2} A_1^\mu + \left(G_2\left(\Delta^2; m, p^2\right) + \frac{\Delta^2 G_1\left(\Delta^2; m, p^2\right)}{\Delta^2 - 4p^2}\right) A_2^\mu\,, \tag{10}$$

where $A_1^\mu = -2i\,\varepsilon^{\mu\,\epsilon\,\epsilon'^*\,P}$ and $A_2^\mu = \frac{2i}{\Delta^2}\,\Delta^\mu\,\varepsilon^{\epsilon\,\epsilon'^*\,P\,\Delta}$. Considering the (anomalous) axial Ward identity, it provides the connection with the matrix element of the divergence of the axial current in Eq. (7) mentioned in the previous paragraph. To be specific, for on-shell gluons, one obtains $G(\Delta^2; m, 0) = D_a(\Delta^2; m, 0) + D_m(\Delta^2; m, 0)$.

We now evaluate A_1^μ and A_2^μ using physical polarization vectors of the gluons. For this analysis, we choose the symmetric reference frame in which

$$P = \left(P^+, \frac{\vec{\Delta}_\perp^2}{8\left(1-\xi^2\right)P^+}, \vec{0}_\perp\right), \qquad \Delta = \left(-2\xi P^+, \frac{\xi\vec{\Delta}_\perp^2}{4\left(1-\xi^2\right)P^+}, \vec{\Delta}_\perp\right), \tag{11}$$

but our general conclusions do not depend on this choice. We use the polarization vectors $\epsilon^\mu_{(1)}$ and $\epsilon^\mu_{(2)}$ specified in Ref. [11]. The linear combinations $\epsilon_{(\pm)} = \mp\left(\epsilon_{(1)} \pm i\,\epsilon_{(2)}\right)/\sqrt{2}$ describe states of definite (light-cone) helicity. With the notation $A^\mu_{1\,(ij)} = -2i\,\varepsilon^{\mu\,\epsilon_{(i)}\,\epsilon'^*_{(j)}\,P}$ *etc.*, we find that, for the specific case of $\mu = +$, the non-zero expressions are

$$A^+_{1\,(++)} - A^+_{1\,(--)}\,, \qquad A^+_{2\,(+-)} - A^+_{2\,(-+)}\,. \tag{12}$$

Due to angular momentum conservation, a gluon helicity flip is forbidden for forward kinematics, implying for on-shell gluons the constraint $\Gamma_5^+(\xi, \vec{\Delta}_\perp = \vec{0}_\perp)\big|_{\text{real}} = 0$. Since, according to Eq. (12), A_2^+ can be non-zero (for $\xi \neq 0$), this constraint means that the form factor $G(0)$ must vanish, which it actually does for $m \neq 0$. Keeping the quark mass is therefore necessary to ensure the conservation of angular momentum. In the following, we will see the same result in the context of the non-local current.

4. Generalized parton distributions

In this section, we revisit the light-cone operator for the non-local axial current, now evaluated between gluon states with different momenta and for on-shell gluons. This means that we consider the correlator

$$F^{[\gamma^+\gamma_5]}_{\lambda\lambda'}(x,\Delta)=\int \frac{\mathrm{d}z^-}{4\pi}\mathrm{e}^{ik\cdot z}\langle g\left(p',\lambda'\right)|\bar{q}\left(-\tfrac{z}{2}\right)\gamma^+\gamma_5\,\mathcal{W}\left(-\tfrac{z}{2},\tfrac{z}{2}\right)q\left(\tfrac{z}{2}\right)|g(p,\lambda)\rangle\Big|_{\mathrm{lc}}\,, \tag{13}$$

which defines two independent generalized parton distributions (GPDs), H_1 and H_2. These GPDs can be calculated according to

$$H_1\left(x,\xi,\Delta^2\right) = \frac{1}{2\left(1-\xi^2\right)}\left(F^{[\gamma^+\gamma_5]}_{++}(x,\Delta)-F^{[\gamma^+\gamma_5]}_{--}(x,\Delta)\right), \tag{14}$$

$$H_2\left(x,\xi,\Delta^2\right) = -\frac{1}{2\xi}\left(F^{[\gamma^+\gamma_5]}_{+-}(x,\Delta)-F^{[\gamma^+\gamma_5]}_{-+}(x,\Delta)\right). \tag{15}$$

Therefore, H_1 is associated with helicity-conserving transitions, while H_2 requires a helicity flip. However, for $\vec{\Delta}_\perp=\vec{0}_\perp$, a gluon helicity flip is forbidden by angular momentum conservation, so H_2 must vanish in this limit [10].

Integrating the GPD correlator over x yields the local current. Comparing with Eq. (9), we find $\int_{-1}^{1}\mathrm{d}x\,H_2(x,\xi,\Delta^2)=G(\Delta^2)$. Thus, H_2 is related to the axial anomaly as already pointed out previously [8, 9].

To compute the GPDs in perturbative QCD, we evaluate the same Feynman diagrams that contribute to the PDF g_1, now for off-forward kinematics. The full results for H_1 and H_2 are lengthy and can be found in Ref. [10]. We emphasize that for $\tau\to 0$ (finite quark mass), H_2 indeed vanishes as discussed above. We repeat that this result is required by the conservation of angular momentum. On the other hand, for $\tau\to\infty$, we find

$$H_2\left(x,\xi,\Delta^2;m\right)=\frac{\alpha_{\mathrm{s}}}{4\pi}\begin{cases}-\dfrac{2(1-x)}{1-\xi^2} & \text{for } \xi\le x\le 1\,,\\ -\dfrac{2}{1+\xi} & \text{for } -\xi\le x\le\xi\,,\end{cases} \tag{16}$$

which agrees with Ref. [9], where the quark mass was neglected right from the start of the calculation. Since the result in Eq. (16) does not depend on Δ^2, it is tempting to conclude that H_2 is finite in the forward limit. However, $\tau\to\infty$ means that Δ^2 must be finite. Put differently, the result in Eq. (16) (for massless quarks) holds for non-zero momentum transfer only. As emphasized in the previous section, the combination of the anomaly and the quark mass is necessary for consistent results in the forward limit of the local current. Likewise, for the non-local axial current, one only finds a meaningful result in the forward limit for a non-zero quark mass.

5. Conclusions

We studied the matrix elements of the local and non-local axial current evaluated between gluon states at $\mathcal{O}(\alpha_s)$ in perturbative QCD, including quark mass effects. We confirmed the connection of the results with the axial anomaly for on-shell and off-shell gluons. In the former case, the quark mass not only regulates the collinear limit, but even leads to a cancellation of terms that would otherwise violate angular momentum conservation. Our results also imply a corresponding cancellation for the box diagram in DVCS, in contrast to a recently reported 'pole' in the forward limit.

The work of I.C. was supported by the National Science Foundation under grant No. PHY-2110472, and by a Summer Research Grant from Temple University. The work of A.F. was supported by the U.S. Department of Energy contract No. DE-AC05-06OR23177, under which Jefferson Science Associates, LLC operates Jefferson Lab. The work of A.M. was supported by the National Science Foundation under grant No. PHY-2110472, and by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under the Quark–Gluon Tomography (QGT) Topical Collaboration with Award DE-SC0023646. The work of S.R. was supported by the German Science Foundation (DFG), grant No. 409651613 (Research Unit FOR 2926), subproject 430915355.

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