MODIFIED HOMOTOPIC APPROACH FOR DIFFRACTIVE PRODUCTION*

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We review the recent developments in the use of the homotopy method for solving the non-linear evolution equation for the diffractive production in deep-inelastic scattering. We introduce part of the non-linear corrections in the linear term. This simplified non-linear evolution equation is solved analytically taking into account the initial and boundary conditions for the process. It turns out that these corrections are rather small and can be estimated in the regular iterative procedure.

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1. Introduction

In this paper, we present a procedure to solve the non-linear equations for a diffractive process that in QCD govern the dynamics in the saturation region. In our previous paper [1, 2], we found a solution of the Balitsky–Kovchegov (BK) equation [3, 4] that gives the dipole scattering amplitude using the homotopy approach. It has been shown that this approach allows us to collect all essential contribution into the linear equation which can be solved analytically, and to propose an iteration procedure, which is partly numerical and leads to small corrections.

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For diffractive production, we have several different kinematic regions for $N^{\rm D}$, where $\sigma^{\rm diff}_{\rm dipole}(r_\perp, Y, Y_0) = \int \mathrm{d}^2 b\, N^{\rm D}(r_\perp, Y, Y_0; \vec{b}\,)$ is the cross section of diffractive production with the rapidity gap larger than Y_0. The non-linear evolution equation for $N^{\rm D}(Y, Y_0, r_{10}; b)$ that describes this diffraction production in deep inelastic scattering has been derived in Ref. [5] and has the following form:

$$\frac{\partial N^{\rm D}(Y, Y_0, r_{10}; b)}{\partial Y} = \frac{\bar{\alpha}_{\rm S}}{2\pi} \int \mathrm{d}^2 r_2 \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \times \left\{ N^{\rm D}_{12} + N^{\rm D}_{20} - N^{\rm D}_{10} + N^{\rm D}_{12} N^{\rm D}_{20} - 2N^{\rm D}_{12} N_{20} - 2N_{12} N^{\rm D}_{20} + 2N_{12} N_{20} \right\}\,, \quad (1)$$

where $N_{ik} = N(Y; r_{ik}; b)$ is the elastic scattering amplitude of a dipole with size r_{ik} and rapidity Y, and $N^{\rm D}_{ik} = N^{\rm D}(Y; Y_0, r_{ik}; b)$ is the cross section of the diffractive production with the rapidity gap larger than Y_0 at the impact parameter b for the same scattering process.

In general, the homotopy method can be used as an effective procedure for solving an equation of the form $\mathcal{L}[u] + \mathcal{N}_{\mathcal{L}}[u] = 0$, where the linear part $\mathcal{L}[u]$ is a differential or integral-differential operator, and the non-linear part $\mathcal{N}_{\mathcal{L}}[u]$ has an arbitrary form. To solve it, we introduce the following equation for the homotopy function $\mathcal{H}(p, u)$: $\mathcal{H}(p, u) = \mathcal{L}[u_p] + p\mathcal{N}_{\mathcal{L}}[u_p] = 0$. Solving this expression, we reconstruct the function

$$u_p\left(Y, \vec{x}_{10}, \vec{b}\,\right) = u_0\left(Y, \vec{x}_{10}, \vec{b}\,\right) + p\, u_1\left(Y, \vec{x}_{10}, \vec{b}\,\right) + p^2\, u_2\left(Y, \vec{x}_{10}, \vec{b}\,\right) + \ldots \quad (2)$$

with $\mathcal{L}[u_0] = 0$. Equation (2) gives the solution to the non-linear equation for $p = 1$. The hope is that several terms in this series will give a good approximation to the solution of the non-linear equation.

The linear equation is obvious in the perturbative QCD region (see Fig. 1), where it is the BFKL equation. However, we will show that inside the saturation region (see Fig. 1), we can find the linearized equation based on the approach of Ref. [6]. We demonstrate that the non-linear term, which includes the remains of the non-linear corrections, leads to small contributions and can be treated in the perturbation approach. This aspect has been discussed in Ref. [2], whose results are here briefly described.

The kinematic region where we are looking for the solution for the scattering amplitude of a dipole ($x_{10} = r$) with a nucleus target in the plot with $\xi_{\rm s}$ and ξ axes is shown in Fig. 1, where $\xi = \ln(x_{10}^2\, Q_{\rm s}^2(Y = Y_A, b))$ and $\xi_{\rm s} = \ln(Q_{\rm s}^2(Y)/Q_{\rm s}^2(Y = Y_A, b))$. One can see that for $z < 0$ we have the perturbative QCD region where the non-linear corrections are small and we can safely use the BFKL linear equation for the scattering amplitude. The geometrical scaling variable z is defined as $z = \ln(r^2\, Q_{\rm s}^2(\delta\tilde{Y}, b)) = \xi_{\rm s} + \xi$, with the saturation moment $Q_{\rm s}^2(Y) = Q_{\rm s}^2(Y_A)\, \mathrm{e}^{\bar{\alpha}_{\rm S}\,\kappa(Y - Y_A)}$, $\kappa = 4.88$, and Y_A denotes $Y_A = \ln A^{1/3}$, where A is the number of nucleons in a nucleus.

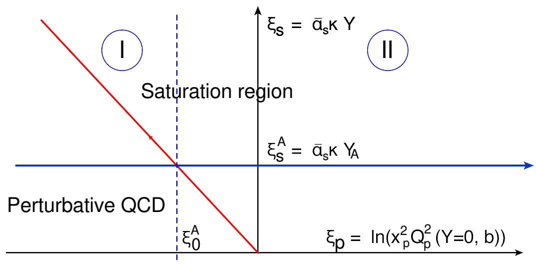

Fig. 1. Saturation region of QCD for the elastic scattering amplitude with $\kappa = \frac{\chi(\gamma_{\rm cr})}{1-\gamma_{\rm cr}}$ and $\chi(\gamma)$ is the BFKL characteristic function. The critical line ($z = 0$) is shown in red. The initial condition for scattering is given at $\xi_{\rm s} = 0$ and for heavy nuclei, the initial condition is shown by the blue line. At the vertical blue dotted line $\xi = \xi_0^A$, the amplitudes described in regions I and II are matched.

For $z > 0$, the non-linear corrections become essential and we enter the saturation region. For the scattering with nuclei, the saturation region can be divided into two parts. For $\xi < \xi_0^A$, the amplitude has the geometric scaling behaviour [1] and depends only on one variable, z. For $\xi > \xi_0^A$, this geometric scaling behaviour is broken. This process can be characterized by two kinematic regions: for $r_\perp Q_{\rm s}(Y_0) < 1$ and $r_\perp Q_{\rm s}(Y - Y_0) < 1$, we can replace N by the BFKL Pomeron. For $r_\perp Q_{\rm s}(Y_0) > 1$ and $r_\perp Q_{\rm s}(Y - Y_0) < 1$, the elastic amplitude is in the saturation region and the production of gluons can be computed using the BFKL Pomeron exchange. Finally, $r_\perp Q_{\rm s}(Y_0) > 1$ and $r_\perp Q_{\rm s}(Y - Y_0) > 1$ is the kinematic region where non-linear corrections for gluon production are essential.

It turns out that Eq. (1) can be rewritten in a simple form introducing a new function,

$$\mathcal{N}\left(z, \delta\tilde{Y}, \delta Y_0\right) = 2N\left(z, \delta\tilde{Y}\right) - N^{\rm D}\left(z, \delta\tilde{Y}, \delta Y_0\right) , \tag{3}$$

where $N(z, \delta\tilde{Y})$ is the solution of the BK equation. The new variables $\delta\tilde{Y}$ and δY_0 are defined as $\delta\tilde{Y} = \bar{\alpha}_{\rm S}(Y - Y_A)$ and $\delta Y_0 = \bar{\alpha}_{\rm S}(Y_0 - Y_A)$. This function has clear physics meaning: the inelastic cross section of all events with a rapidity gap from $Y = 0$ to $Y = Y_0$. Then, Eq. (1) in terms of the function $\mathcal{N}$ takes the form of the BK equation, *viz.*

$$\frac{\partial \mathcal{N}_{01}}{\partial Y} = \bar{\alpha}_{\rm S} \int \frac{{\rm d}^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \left\{\mathcal{N}_{02} + \mathcal{N}_{12} - \mathcal{N}_{02}\mathcal{N}_{12} - \mathcal{N}_{01}\right\} , \tag{4}$$

with the initial conditions $\mathcal{N}(z \to z_0, \delta\tilde{Y} = \delta Y_0, \delta Y_0) = 2N(z_0, \delta Y_0) - N^2(z_0, \delta Y_0)$; it holds only in region I, while in region II, we have to use a more general expressions for the elastic scattering amplitudes (see Ref. [1]).

Our strategy for finding a solution looks as follows: first, we solve Eq. (4) with $\mathcal{N}_{01} = 1 - \Delta^{\rm D}$, and after that we return to Eq. (1).

2. Modified homotopy approach for $\Delta^{\rm D}_0$ and numerical estimation for $\Delta^{\rm D}_1$

Including part of the non-linear term into the definition of $\mathcal{L}[u_0]$, we can find the solution to the non-linear equation of Eq. (4), suggested in Ref. [1]. Following the main ideas of Ref. [6], we solve Eq. (4) replacing $\mathcal{N}(z, \delta Y_0)$ by $1 - \Delta^{\rm D}(z, \delta Y_0)$. For this function, the equation takes the form

$$\frac{\partial \Delta^{\rm D}_{01}}{\partial Y} = \bar{\alpha}_{\rm S} \int \frac{{\rm d}^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \left\{ \Delta^{\rm D}_{02} \Delta^{\rm D}_{12} - \Delta^{\rm D}_{01} \right\} , \tag{5}$$

with the corresponding initial conditions for $\Delta^{\rm D}_{01}$. We suggest to simplify the non-linear term replacing it by

$$\int \frac{{\rm d}^2 x_{02}}{2\pi} \frac{x^2_{01}}{x^2_{02}\, x^2_{12}} \Delta^{\rm D}_{02} \Delta^{\rm D}_{12} \to \Delta^{\rm D}_{01} \int\limits_0^z {\rm d}z' \Delta^{\rm D}_{02} = \Delta^{\rm D}_{01} \left(\zeta - \int\limits_z^\infty {\rm d}z' \Delta^{\rm D}_{02} \right) . \tag{6}$$

This contribution stems from the region $x_{02} \ll x_{01}$ (see Ref. [6]) and, to find the solution $\Delta^{\rm D}_0$, we need to solve the equation $\mathcal{L}(\Delta^{\rm D}_0) = 0$. Introducing $\Delta^D_0(z, \xi_{\rm s}) = \exp(-\Omega^{(0)}(z, \xi_{\rm s}))$, we can obtain the next general equation ($\xi_{\rm s} = \kappa\, \delta\tilde{Y}$),

$$\kappa \frac{\partial^2 \Omega^{(0)}}{\partial \xi_{\rm s}\, \partial z} = 1 - {\rm e}^{-\Omega^{(0)}(\vec{z}, \xi_{\rm s})} , \tag{7}$$

which in region I has the travelling wave solution (Ref. [7]) with the geometric scaling behaviour that can be found from the following implicit equation:

$$\mathcal{U}\left(\Omega^{(0,{\rm I})}, \Omega^{(0)}_0 = a\right) = \int\limits_{\Omega^{(0)}_0}^{\Omega^{(0)}} \frac{{\rm d}\Omega'}{\sqrt{\Omega' + \exp(-\Omega') - \Omega^{(0)}_0}} = \sqrt{\frac{2}{\kappa}} \left(z + C_2\right) , \tag{8}$$

where C_2 as well as $\Omega^{(0)}_0$ can be found using the boundary conditions. We can solve Eq. (8) for $\Omega^{(0,{\rm I})}(z)$ and obtain $\Delta^{(0,{\rm I})}_0$ which is given by

$$\Delta^{(0,{\rm I})}_0(z) = \Delta_{\rm LT}(z) \exp\left(-a + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-1)!!(2\kappa)^{k+\frac{1}{2}}}{2^k k\, k!(z - \tilde{z})^{2\left(k+\frac{1}{2}\right)}} \Delta^k_{\rm LT}(z) \right) , \tag{9}$$

where $\varDelta_{\rm LT}(z) = \exp(-\frac{(z-\tilde{z})^2}{2\,\kappa})$ is the Levin–Tuchin solution [6]. In region II, $\varOmega^{(0,{\rm II})}(z,\xi_{\rm s})$ is described by the equation

$$\frac{\partial^2 \varOmega^{(0,{\rm II})}(z,t)}{\partial z^2} - \frac{\partial^2 \varOmega^{(0,{\rm II})}(z,t)}{\partial t^2} = \frac{1}{\kappa}\left(1 - {\rm e}^{-\varOmega^{(0,{\rm II})}(z,t)}\right)\,, \tag{10}$$

where $z = \xi_{\rm s} + \xi$ and $t = \xi_{\rm s} - \xi$, whose solution is given again by the implicit travelling wave solution $\mathcal{U}(\varOmega^{(0,{\rm II})},\varOmega_0)$ (see Eq. (8)). Therefore, for $\varDelta^{(0,{\rm II})}(z,t)$, we have

$$\varDelta^{(0,{\rm II})} = \varDelta_{\rm LT}\exp\left(-a + \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(2k-1)!!\,(2\kappa)^{k+\frac{1}{2}}}{2^k k\,k!\,((1+\nu)z+\nu t-\hat{z})^{2\left(k+\frac{1}{2}\right)}}\varDelta_{\rm LT}^k\right)\,. \tag{11}$$

Now, we are in condition to use the homotopic approach to obtain the first correction $\varDelta_1^{\rm D}$ from the general equation valid in regions I and II

$$\left(\kappa\frac{\partial}{\partial z} + z\right)\varDelta_1^{\rm D}(z,\xi,z_0) = -\mathcal{N}_{\mathcal{L}}\left[\varDelta_0^{\rm D}\right]\,, \tag{12}$$

where the non-linear contribution is given by

$$\mathcal{N}_{\mathcal{L}}\left[\varDelta^{\rm D}\right] = \bar{\alpha}_{\rm S}\int\frac{{\rm d}^2 x_{02}}{2\pi}\frac{x_{01}^2}{x_{02}^2\,x_{12}^2}\varDelta_0^{\rm D}(x_{02})\varDelta_0^{\rm D}(x_{12}) - \varDelta_0^{\rm D}\int^{x_{01}^2}\frac{{\rm d}x_{02}^2}{x_{02}^2}\varDelta_{02}^{\rm D}\,. \tag{13}$$

For numerical estimation, we have to insert $\varDelta_0^{\rm D}$ in the non-linear term and the particular solution in region I with geometric scaling $\varDelta_1^{\rm D}$ is given by

$$\varDelta_1^{\rm D}(z,z_0) = -\varDelta_0^{\rm D}(z,z_0)\int_z^{\infty}{\rm d}z'\frac{1}{\varDelta_0^{\rm D}(z',z_0)}\mathcal{N}_{\mathcal{L}}\left[\varDelta_0^{\rm D}(z')\right]\,. \tag{14}$$

In Fig. 2, we present the numerical estimates for $\mathcal{N}_{\mathcal{L}}$ and the ratio $\varDelta_1^{\rm D}/\varDelta_0^{\rm D}$ turns out to be small. In region II $\varDelta_1^{\rm D}(z,\xi,z_0)$ has been found in our previous paper [1] (see Eqs. (58)–(67)) and $\mathcal{N}_{\mathcal{L}}[\varDelta^{\rm D}]$ has the same form as Eq. (13) with $\varDelta_0^{(0,{\rm II})}$. Finally, the solution $\varDelta_1^{\rm D}(z,\xi,z_0)$ has the following form:

$$\varDelta_1^{\rm D}(z,\xi,z_0) \sim \varDelta_0^{\rm D}(z,\xi,z_0)\left(-{\rm erf}\left(\frac{z_0-4\ln 2}{\sqrt{2\kappa}}\right) + {\rm erf}\left(\frac{z-4\ln 2}{\sqrt{2\kappa}}\right)\right)\,, \tag{15}$$

which vanishes at $z\to z_0$ and in the region of large z, this contribution is rather small.

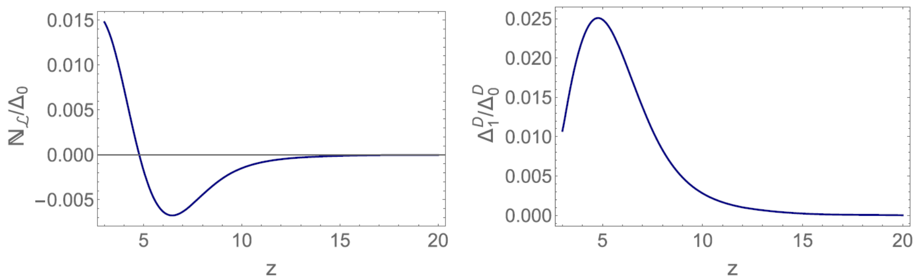

Fig. 2. Left describes the ratio $\mathcal{N}_\mathcal{L}[\Delta_0^{\rm D}]/\Delta_0^{\rm D}$. Right describes $\Delta_1^{\rm D}/\Delta_0^{\rm D}$ *versus* z for $z > z_0$. z_0 is taken to be 3, $\bar{\alpha}_{\rm S} = 0.2$.

3. Conclusions

Using the modified homotopy approach, we found that the first iteration of the homotopy approach gives the main contribution in both kinematic regions which we consider for the diffractive production. We also found that for $\xi < \xi_0^A$, our solution shows the geometric scaling behaviour, while for $\xi > \xi_0^A$, this behaviour is strongly violated. We found that the analytical solution of the non-linear equation reproduces the initial and boundary conditions. The second iteration with zero initial and boundary conditions turns out to be small and could be taken into account together with higher iterations using the regular perturbative procedure.

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