TRIPLE (AND QUADRUPLE) SOFT-PARTON RADIATION IN QCD HARD SCATTERING*

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We compute the tree-level current for triple soft-gluon emission from both massless and massive hard partons. The three-gluon current is expressed in terms of the maximally non-Abelian irreducible correlations. We compute the soft behaviour of squared amplitudes and the colour correlations produced by the squared current. The radiation of one and two soft gluons leads to colour dipole correlations. Triple soft-gluon radiation produces, in addition, colour quadrupole correlations between the hard partons. We examine the soft and collinear singularities of the squared current in various energy-ordered and angular-ordered regions. Considering triple soft-gluon radiation from three hard partons, colour quadrupole interactions break the Casimir scaling symmetry between quarks and gluons. We also present some results on the radiation of four soft gluons from two hard partons, and we discuss the colour monster contribution and its relation with the violation (and generalization) of Casimir scaling. We also compute the first correction of $(1/N_c^2)$ to the eikonal formula for multiple soft-gluon radiation with strong energy ordering from two hard gluons. Finally, we compute also the tree-level current and its squared modulus for emission of soft quark–antiquark–gluon states.

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1. Introduction

Our aim is to investigate the behaviour of a scattering amplitude when some external gluons become soft. When massless partons $\{q_i\}$ become soft with respect to hard partons $\{p_k\}$, the amplitude $M(\{p_k\},\{q_i\})$ diverges, and the leading divergence can be factorized as a singular soft current operator $J(\{q_i\})$ acting on the hard amplitude $M(\{p_k\})$ with the soft partons removed.

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Here, we present detailed results for the emission of three (and four) soft gluons [1]. Our results of the emission of soft gluon–quark–antiquark triplets can be found in [2]. They agree with those of [3].

2. Soft currents

Let us first consider the case of one gluon emission in QCD [4] in terms of the non-Abelian colour matrices

$$J^a(q) = J^{a,\mu}(q)\varepsilon_\mu(q) = \sum_{k=1}^{K} \frac{g T_k^a p_k^\mu}{p_k \cdot q}\varepsilon_\mu(q) \qquad \begin{cases} (T_q^a)_{bc} = t^a_{bc} \\ (T_{\bar{q}}^a)_{bc} = -t^a_{cb} \\ (T_g^a)_{bc} = i f^{abc} \end{cases} . \tag{1}$$

For physical amplitudes, with overall zero charge (colour singlets), the currents are conserved[1]: $\sum_{k=1}^K T_k^a \overset{\text{cs}}{=} 0$ hence $q_\mu J^{a,\mu} \overset{\text{cs}}{=} 0$.

Also, the current for two soft-gluon emission has been known for long time [5]. It can be conveniently expressed as the symmetric product of two 1-gluon currents, plus a remainder $\Gamma(1,2)$ which represents the maximally non-Abelian correlation between the two soft gluons

$$\begin{aligned} J^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) &= J^{a_1}_{\mu_1}(q_1) J^{a_2}_{\mu_2}(q_2) + \Gamma^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) \qquad \left(A\,B \equiv \frac{1}{2}(AB+BA)\right), \\ \Gamma^{a_1 a_2}_{\mu_1\mu_2}(q_1,q_2) &= \mathrm{i} f^{a_1 a_2 b} \sum_{k\in\text{hard}} T_k^b \, \gamma_k^{\mu_1\mu_2}(q_1,q_2)\,, \\ \gamma_k^{\mu_1\mu_2}(q_1,q_2) &= \frac{1}{p_k\cdot(q_1+q_2)} \left\{ \frac{p_k^{\mu_1} p_k^{\mu_2}}{2\, p_k\cdot q_1} + \frac{1}{q_1\cdot q_2}\left(p_k^{\mu_1} q_1^{\mu_2} + \frac{1}{2} g^{\mu_1\mu_2} p_k\cdot q_2 \right)\right\} \\ &\quad - (1 \leftrightarrow 2)\,. \end{aligned} \tag{2}$$

The colour structure of Γ is just a colour matrix contracted with a structure constant. This current is conserved, if applied on colour-singlet states, even without contracting the remaining Lorentz index with the polarization vector of the second gluon. In the Abelian case, only the independent product of single-gluon currents survives. In QCD however, we have colour correlation also from the symmetric product.

We have evaluated [1] the current for the emission of three gluons computing the relevant Feynman diagrams with gluon insertions on the external lines, where we use eikonal vertices on hard lines and exact vertices and propagators elsewhere. We find that the current, when acting on colour singlets, does not depend on the gauge and is conserved. It can conveniently be

[1] The notation $\overset{\text{cs}}{=}$ means equality of operators when acting on colour singlet states.

expressed as a symmetric product of three 1-gluon currents, the symmetric product of one 1-gluon current and the 2-gluon non-Abelian term, plus an irreducible, maximally non-Abelian remainder $\mathit{\Gamma}(1,2,3)$:

$$J(1,2,3) \;=\; J(1)\,J(2)\,J(3) + \left[\sum_{\text{cyc.123}} J(1)\,\mathit{\Gamma}(2,3)\right] + \mathit{\Gamma}(1,2,3)\,, \tag{3}$$

$$\mathit{\Gamma}(1,2,3) \;=\; \sum_{k\in\text{hard}} T_k^b \sum_{\text{cyc.123}} \sum_s f^{a_1 a_2\, s} f^{s\, a_3 b}\, \gamma_k^{\mu_1\mu_2\mu_3}(q_1,q_2;q_3)\,. \tag{4}$$

We can present the soft currents also in terms of the decomposition in colour-ordered subamplitudes, according to the results of Berends and Giele [6] in the soft limit.

3. Squared soft currents

The objects that are relevant for cross sections involving some number of soft gluons are the squared currents. By using Dirac's notation [7] in colour and helicity space, we can consider an amplitude which depends on colours and helicities of the outgoing particles as the components of an abstract vector in colour and helicity space. In this space, a soft current is a rectangular matrix. The squared amplitude is obtained by multiplying the amplitude by its complex conjugate, and summing over final-state quantum numbers. On the contrary, the modulus square of the current is a square matrix in the space of the hard partons' quantum numbers. Due to the conservation of the soft gluon currents, the square currents are explicitly gauge-invariant operators when acting on colour singlet states.

The square current for one soft gluon is just a sum of colour dipoles $\boldsymbol{T}_i \cdot \boldsymbol{T}_k \;\equiv\; \sum_a T_i^a T_k^a$ representing the colour matrices of two hard particles summed over the same gluon colour index which connects them

$$|\boldsymbol{J}(q)|^2 \overset{\text{cs}}{=} - \sum_{i,k\in\text{hard}} \boldsymbol{T}_i \cdot \boldsymbol{T}_k\, \mathcal{S}_{ik}(q) =: W(q)\,, \qquad \mathcal{S}_{ik}(q) = \frac{p_i \cdot p_k}{p_i \cdot q\; p_k \cdot q}\,. \tag{5}$$

The square current for two soft gluons can be conveniently expressed as the symmetric product of the square currents of a single gluon, plus an irreducible correlation term $W(q_1,q_2)$

$$|\boldsymbol{J}(q_1,q_2)|^2 \;\overset{\text{cs}}{=}\; W(q_1)\,W(q_2) + W(q_1,q_2)\,, \tag{6}$$

$$W(q_1,q_2) \;=\; -C_A \sum_{i,k\in\text{hard}} \boldsymbol{T}_i \cdot \boldsymbol{T}_k\, \mathcal{S}_{ik}(q_1,q_2)\,. \tag{7}$$

This irreducible correlation is again a sum of colour dipoles, times the adjoint Casimir, and a kinematical coefficient which has been derived in [5].

We computed the square of the 3-soft-gluon current. It is also conveniently expressed as a sum of symmetric products of an object with one or two soft gluons, plus an irreducible correlation $W(1,2,3)$

$$|\boldsymbol{J}(q_1,q_2,q_3)|^2 \overset{\text{cs}}{=} W(q_1)\,W(q_2)\,W(q_3) + \left[\sum_{\text{cyc.123}} W(q_1)\,W(q_2,q_3)\right] + W(q_1,q_2,q_3)\,. \qquad (8)$$

$W(q_1,q_2,q_3)$ involves not only colour dipoles but also colour quadrupoles $Q_{iklm} \equiv \frac{1}{2} f^{ab,cd} \left(T_l^a \left\{T_i^c, T_k^d\right\} T_m^b \ + \ \text{h.c.}\right)$

$$W(q_1,q_2,q_3) = -C_A^2 \sum_{i,k} \boldsymbol{T}_i \cdot \boldsymbol{T}_k \, \mathcal{S}_{ik}(q_1,q_2,q_3) + \sum_{iklm} Q_{iklm} \, \mathcal{S}_{iklm}(q_1,q_2,q_3)\,. \qquad (9)$$

The kinematical coefficients of the dipole and quadrupole terms are rather cumbersome, and can be read in Ref. [1]. They considerably simplify in the case of strong energy ordering (SEO) of the soft gluons. In particular, the dipole kinematical coefficient is remarkably symmetric with respect to the permutations of the three soft momenta, while the kinematical coefficient of the quadrupole is remarkably symmetric in the exchange of the two softest momenta.

We consider now the collinear singularities of the squared currents. In the case of one soft gluon collinear to parton B, it is easy to show the absence of colour correlations due to colour coherence, and the soft current is proportional to the Casimir of the B hard parton.

In the case of three soft gluons, the expansion in irreducible correlations reduces the collinear singularities of Ws. $W(2,3)$ is singular in the double-collinear limit of the two soft gluons (according to the exact $P^{\mu\nu}_{g_1 g_2}$) and in the triple-collinear limit of the two soft gluons and a hard parton.

$W(1,2,3)_{\text{dipole}}$ is singular in the double-collinear limit of two soft gluons (reproducing known exact results), in the triple-collinear limit of the three soft gluons (according to the exact $P^{\mu\nu}_{g_1 g_2 g_3}$), and in the quadruple-collinear limit of the three soft gluons and a hard parton (predicting soft $P^{ss'}_{g_1 g_2 g_3 C}$).

$W(1,2,3)_{\text{quadrupole}}$ has no collinear singularity at all.

4. Three hard partons

We now consider the special case of amplitudes with three hard partons $|ABC\rangle$ plus soft gluons, and possibly other colourless particles. Due to flavour conservation, we can have only three partonic configurations: In the case of $|gq\bar{q}\rangle$, there is only one colour-singlet state. Therefore, the squared current, which conserves colour, acts on a one-dimensional state, so that the soft factorization becomes a multiplication by a c-number, the eigenvalue of the soft current on this state. This fact is valid to all perturbative orders in the amplitudes and in the current.

In the case of three hard gluons, we can have two distinct colour-singlet states: the colour-antisymmetic one, where the gluons are in an $|f^{abc}\rangle$ state, and the colour-symmetric $|d^{abc}\rangle$ state. Since they have opposite charge conjugation, and the soft current conserves charge conjugation, it turns out that the square current is a 2×2 diagonal matrix in the basis of these two states. The eigenvalues on these states are equal for one and two soft gluons, but differ for three or more soft gluons due to the quadrupole contribution, which vanishes on the $|d\rangle$ state but not on the $|f\rangle$ state.

The singlet state of three hard partons are eigenstates of all dipole operators, whose eigenvalue is a linear combination of Casimir coefficients. Therefore, one- and two-gluon square currents are c-numbers containing the Casimir of particle A (always a gluon) and of particle B, which can be either a quark or a gluon. These currents obey the so-called Casimir scaling: changing $|gq\bar{q}\rangle$ to $|ggg\rangle$ amounts to replace C_F with C_A.

The same holds for the dipole terms of the three-gluon square current. However, the quadrupole term violates Casimir scaling due to its peculiar action on the three different hard states. Note that when increasing the number of colours, the dipole terms grow like N_c^3, while the quadrupole ones just like N_c. Another peculiarity of the quadrupole term is that its kinematical coefficient is collinear safe with respect to angular integration over soft-gluon momenta.

We have also shown that, when three hard gluons emit N soft gluons with strong energy ordering $E_1 \ll E_2 \ll \ldots E_N$, the squared current, to leading order in the number of colours, has a very simple form, in terms of a multi-eikonal function F_{eik} introduced in Ref. [4].

5. Two hard partons

We consider soft-gluon emission from two hard partons in a colour-singlet state $|BC\rangle$. There are just two such states: $|q\bar{q}\rangle$ and $|gg\rangle$. In both cases, the colour space is one-dimensional and we have again c-number factorisation. Non-Abelian effects are in $\mathrm{SU}(N_c)$ colour coefficients. The eigenvalues of the squared current on such states obey Casimir scaling up to three soft gluons.

In the case of emission of N soft gluons from two hard gluons, we have checked the Bassetto–Ciafaloni–Marchesini formula [4] in terms of the multi-eikonal function F_{eik}, up to colour suppressed contributions. Actually, in the case of four soft gluons, we can explicitly compute the corresponding correction, because a four-soft current is constrained by a three-soft current.

It turns out that the irreducible correlation for four soft gluons, in the limit of $E_4 \gg E_{1,2,3}$, has a dipole contribution and a quadrupole contribution. Therefore, a term satisfies Casimir scaling, while the other does not

$$W(q_1 \cdots q_4)|_{BC} = C_B \left[C_A^3 \, w_{BC}^{(\mathrm{L})}(q_1 \cdots q_4) + \lambda_B N_c \, w_{BC}^{(\mathrm{S})}(q_1 \cdots q_4) \right] , \quad (10)$$

with $\lambda_F = 1/2$ on a $|gq\bar{q}\rangle$ state, $\lambda_A = 3$ on an $|f\rangle$ state, and $\lambda = 0$ on an $|d\rangle$ state. The quadrupole term that violates Casimir scaling is related to the quartic Casimir. Therefore, changing the two hard particles from $q\bar{q}$ to gg can be taken into account by a generalized Casimir scaling where, in addition to change the quadratic Casimir, we change also the quartic one. In the case of strong ordering, the kinematical coefficients can be derived from the three-gluon current, and we have found the first correction to the BCM eikonal formula.

Reference [8] examined the 4-soft-gluon radiation from two massless hard partons in strong energy ordering, and found a contribution proportional to $C_B \, N_c$ from the so-called *colour monster* diagram. Our results are fully consistent with the colour monster, and are related to the quartic Casimir: $C_B \, \lambda_B \, N_c = 2 \, \frac{d_{AB}^{(4)}}{D_B} - \frac{1}{12} \, C_B \, C_A^3$. In particular, the collinear singularities of the subleading coefficient $w_{BC}^{(\mathrm{S})}$ contribute, at the inclusive level, to the soft limit $\propto \alpha_{\mathrm{S}}^4 \, d_{AB}^{(4)} \, / \, \epsilon$ of the collinear evolution kernel of the parton distribution functions. The soft limit of the evolution kernel is proportional to the cusp anomalous dimension that violates Casimir scaling.

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