ILUMI4d ALGORITHM TO COMPUTE THE LUMINOSITY AND SPACE-TIME STRUCTURE OF LUMINOSITY AT A COLLIDER*

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ILUMI4d, an algorithm to compute the single-bunch luminosity and the space-time structure of luminosity at a collider for arbitrary beam parameter settings, is described. The performance of ILUMI4d is benchmarked for different beam settings (no crab angle), by comparing the computed luminosity for Gaussian bunches with the analytical formula (no hourglass effect) and the numerical integrator (hourglass effect) for the standard LHC settings. The results obtained with ILUMI4d show an agreement better than 1.2%. Furthermore, the vertex distribution as a function of the bunch crossing time is computed for an ideal and full compensating crab cavity.

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1. Introduction

The general formula [1] to compute the luminosity of a single-bunch crossing at a collider L_{sb} is given by

$$L_{\mathrm{sb}} = \int \mathrm{d}t\, \mathrm{d}^3\boldsymbol{x}\, \rho_1\rho_2\, K\,; \qquad K = \sqrt{(\vec{v}_1 - \vec{v}_2)^2 - \frac{(\vec{v}_1 \times \vec{v}_2)^2}{c^2}}\,, \tag{1}$$

$$\int \mathrm{d}^3\boldsymbol{x}\, \rho_i(\boldsymbol{x},t) = N_i\,, \qquad i = 1,2\,. \tag{2}$$

where $\rho_1(\boldsymbol{x},t)$ and $\rho_2(\boldsymbol{x},t)$ are the normalised densities of the N_1, N_2 bunch particles that move with speed $\vec{v_1}$ and $\vec{v_2}$ in the lab frame, and K is the kinematic factor. For Gaussian distributions, the luminosity of a single-bunch crossing can be computed with the analytical formula [2, 7], when the hourglass effect is not taken into account. With the standard numerical integrator [4], the bunch crossing luminosity can be computed for defined time steps, when the hourglass effect is taken into account. ILUMI4d uses

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concepts equivalent to the ones used in [5, 6] and it allows to compute the single-bunch luminosity, the region of luminosity and the vertex footprints of the interaction points as function of the bunch crossing time on a statistical level. This information is the key for the simulation and understanding of timing detectors at colliders, which aim to measure by time of flight (TOF) the production time of the vertices relative to a reference bunch clock system to separate vertices by their time tag.

2. ILUMI4d algorithm

2.1. Introduction

The proton bunch is simulated as a vectorised point cloud with $M = 10^4$ elements representing an unbiased statistical sample of the real proton distribution forming the single bunch with $N \sim 10^{11}$ protons and $D = N/M$ being the sample reduction factor. The two bunches are moving independently in relation to each other on individual trajectories with speed c. At each of N_{fr} equidistant space points per bunch $\mathrm{SP}_{1,i}(x_{1,i}, y_{1,i}, z_{1,i})$, $\mathrm{SP}_{2,i}(x_{2,i}, y_{2,i}, z_{2,i})$, with $(i = 0, \ldots, N_{\mathrm{fr}} - 1)$ along the trajectories, the overlap volume of the two-point clouds is voxelised with rectangular cuboid voxels in the fixed lab coordinate system and the convolution is performed to compute both, the luminosity and the produced number of interactions for a given production cross section for each voxel.

2.2. Coordinate systems

To process the mathematical convolution operations, the bunch collision is expressed in different coordinate systems. In Fig. 1, left, there are described the experiment reference system O (collision plane x–z), the refer-

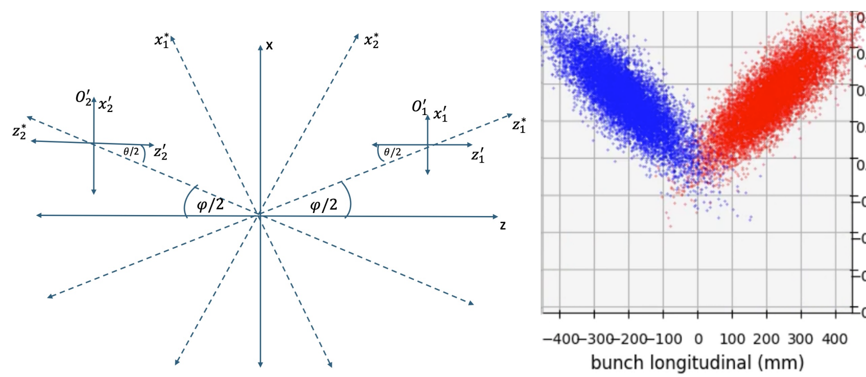

Fig. 1. IlUMI4d coordinate system and generated 3d point clouds of bunch1 and bunch2, $\phi/2 = 150\ \mu$rad, $\theta = 0$, $\epsilon_{x,y} = 3.17\times10^{-10}$ m, $\sigma_z = 90$ mm, $\sigma_{x,y} = 9.7\ \mu$m, $\beta^*{}_{x,y} = 0.3$ m.

ence systems of bunch1 and bunch2 (O_1^*, O_2^*) rotated by the crossing angle ($\varphi/2$), and the moving local reference systems of bunch1 and bunch2 ($O_1^{'}$, O'_2) rotated by the crab angle ($\theta/2$) at every space point $\mathrm{SP}_{1/2,i}$. With linear operators (3d rotation matrix, linear shift), the coordinate of each element of the point cloud can be computed in any of the other coordinate systems. The operators $T_{1,i}$ and $T_{2,i}$ transform at each space point $\mathrm{SP}_{1/2,i}$ the local cloud point coordinates $(x_{1,ik}^{'}, y_{1,ik}^{'}, z_{1,ik}^{'})$, $(x_{2,ik}^{'}, y_{2,ik}^{'}, z_{2,ik}^{'})$ to the corresponding lab coordinates $(x_{1,ik}, y_{1,ik}, z_{1,ik})$, $(x_{2,ik}, y_{2,ik}, z_{2,ik})$ (Fig. 1, right).

2.3. Gaussian bunch shapes and point clouds

The Gaussian bunch profiles are defined by the bunch length σ_z, the transverse beam size σ_x and σ_y. $\beta_{x,y}^*$ is the beta function at the interaction point and $\epsilon_{x,y}$ is the transverse emittance [2, 3]

$$\sigma_{x,y}(z) = \sqrt{\beta_{x,y}^* \left(1 + \left(\frac{z}{\beta_{x,y}^*}\right)^2\right) \epsilon_{x,y}} \,. \tag{3}$$

If the bunch length with respect to β^* is small, the transverse beam size is constant in the collision region and $\sigma_{x,y}(z) = \sigma_{x,y}^* = \sqrt{\beta_{x,y}^* \epsilon_{x,y}}$. Only in this case the analytical formulas allow us to compute the luminosity of the single-bunch crossing. However, in many realistic operation scenarios of colliders, β^* is smaller than the bunch length and, consequently, $\sigma_{x,y} \neq$ constant, and Eq. (3) applies. In the literature, this effect is called *hourglass effect*, as the bunch shape is deformed like an hourglass, when it passes the interaction region. In this case, the luminosity can only be computed with numerical integration methods. In ILUMI4d, the hourglass effect can be switched on and off. Equations (4), (5) and (6), (7) are used to compute the Gaussian point cloud distributions in the local bunch coordinate system for the two cases:

2.3.1. Hourglass effect off

$$\rho_k^{'} = \frac{N}{(2\pi)^{\frac{3}{2}} \sigma_{xk}\sigma_{yk}\sigma_{zk}} \, \mathrm{e}^{-\frac{1}{2}\left(\frac{x'^2}{\sigma_{xk}^2} + \frac{y'^2}{\sigma_{yk}^2} + \frac{z'^2}{\sigma_{zk}^2}\right)} , \tag{4}$$

$$\sigma_{\xi k} = \sqrt{\beta_{\xi k}^* \, \epsilon_{\xi k}} \,, \qquad \xi = x, y \,, \qquad k = 1, 2 \,, \tag{5}$$

2.3.2. Hourglass effect on

$$\rho_{ki}^{'} = \frac{N}{(2\pi)^{\frac{3}{2}} \sigma_{xk}(z_{ki})\sigma_{yk}(z_{ki})\sigma_{zk}} \, \mathrm{e}^{-\frac{1}{2}\left(\frac{x'^2_{ki}}{\sigma_{xk}^2(z_{ki})} + \frac{y'^2_{ki}}{\sigma_{yk}^2(z_{ki})} + \frac{z'^2_{ki}}{\sigma_{zk}^2}\right)} , \tag{6}$$

$$\sigma_{\xi k}(z_{ki}) = \sqrt{\beta^*_{\xi k}\left(1+\left(\frac{z_{ki}}{\beta^*_{\xi k}}\right)^2\right)\epsilon_{\xi k}}\,, \qquad \xi = x, y\,, \qquad k = 1, 2\,, \tag{7}$$

$$z_{ki} = T_{ki}\, z^{'}_{ki}\,, \qquad i = 1, \ldots, N_{\rm fr}\,. \tag{8}$$

The hourglass effect is demonstrated for a Gaussian point cloud in Fig. 2, right with unrealistic small $\beta^* = 0.03$ m.

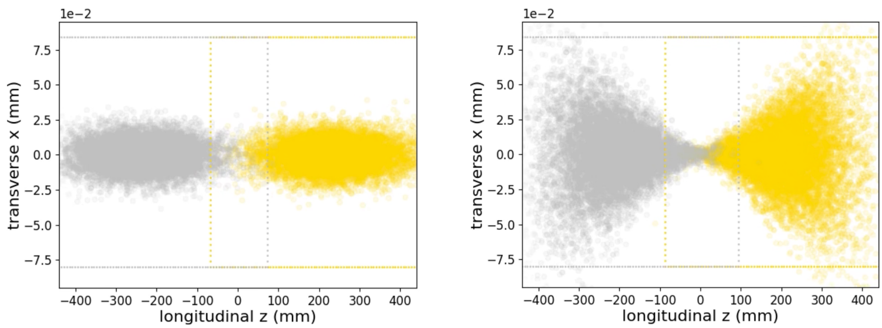

Fig. 2. bunch1 and bunch2 hourglass off (left): $\phi = 0$, $\theta = 0$, $\epsilon_{x,y} = 3.17 \times 10^{-10}$ m, $\sigma_z = 90$ mm, $\sigma_{x,y} = 9.7$ μm, and $\beta^*{}_{x,y} = 0.3$ m; hourglass on (right): $\beta^*{}_{x,y} = 0.03$ m

2.4. Simulation of bunch crossing

At equidistant space points, with a spacing of $c\,\Delta t$ along the axis z^*_1 and z^*_2, the point clouds of bunch1 and bunch2 are computed in the local bunch coordinate systems $O^{'}_1$ and $O^{'}_2$. The distance of the space points for this simulation is chosen to be $\sim$ 15 ps for a bunch length of $\sim$ 1.2 ns. A total $N_{\rm fr} = 131$ time frames (with time frames index $[-\frac{N_{\rm fr}-1}{2}, 0, +\frac{N_{\rm fr}-1}{2}]$) are generated to cover the full collision process with the center of the two bunches at $i = 0$ located at $(z = 0)$ in the lab coordinate system.

2.5. Simulation of collision process and computation of luminosity

For each time frame, the overlap region of the two bunches is voxelised in the lab coordinate system and the luminosity is computed according to Eqs. (1) and (9). The total of $N_{\rm vox} = \bar{N}^3_{\rm vox}$ geometric voxels, ($\bar{N}_{\rm vox} = 40$), covering the interaction volume region, are in this process converted to lumi voxels with the lumi value (physical unit of $\frac{1}{\rm m^2}$) assigned to the grid point of the respective geometric voxels. By multiplying the luminosity value of a lumi voxel with the production cross section, the probability for a vertex production is computed and the lumi voxels are converted to vertex voxels. The vertex voxel distribution is interpreted as $\mathrm{PDF}_{\rm vertex}$ for each time frame with the corresponding $\mathrm{CDF}_{\mathrm{vertex}}$ [8]. With standard Monte Carlo methods, the realisation of a vertex is computed based on the $\mathrm{CDF}_{\mathrm{vertex}}$ and the vertex coordinates are distributed with equal probability within the voxel volume. The sum of all vertex voxel values in all time frames results in the number of vertices produced in the single-bunch collision. The bunch luminosity in ILUMI4d is computed with

$$L_{\mathrm{sc}}^{4\mathrm{d}} = \sum_{i=-\frac{N_{\mathrm{fr}}-1}{2}}^{\frac{N_{\mathrm{fr}}-1}{2}} \sum_{j,k,l=1}^{\bar{N}_{\mathrm{vox}}} D^{12} \left(V_{j,k,l}^{\mathrm{exp}} \odot \rho_{j,k,l,i}^{1\mathrm{exp}} \odot \rho_{j,k,l,i}^{2\mathrm{exp}} \right) K\, \Delta t\,. \tag{9}$$

$V_{j,k,l}^{\mathrm{exp}}$ is the volume of the rectangular cuboid, $\rho_{j,k,l,i}^{1\mathrm{exp}}$ and $\rho_{j,k,l,i}^{2\mathrm{exp}}$ are the densities of the bunch1 and bunch2 cloud points in the voxel, and $\odot$ indicates the Hadamar product operator. The factor $D^{12} = (\frac{N_1\, N_2}{D_1\, D_2})$ is the factor to scale up from the point clouds of bunch1 and bunch2 to the real bunch population, K is the kinematic factor, and Δt is the time between two space points.

$\mathcal{LUMI}_{j,k,l,i}^{\mathrm{exp}}$ is the luminosity voxel distribution for a single-time frame i,

$$\mathcal{LUMI}_{j,k,l,i}^{\mathrm{exp}} = D^{12} \left(V_{j,k,l}^{\mathrm{exp}} \odot \rho_{j,k,l,i}^{1\mathrm{exp}} \odot \rho_{j,k,l,i}^{2\mathrm{exp}} \right) K\, \Delta t\,, \tag{10}$$

and μ_i are the vertices produced (pileup) for a time frame i,

$$\mu_i = \sum_{j,k,l=1}^{\bar{N}_{\mathrm{vox}}} \mathcal{LUMI}_{j,k,l,i}^{\mathrm{exp}}\, \sigma_{\mathrm{production}}\,, \tag{11}$$

$\mathrm{vertex}_{j,k,l,i}$ is the PDF of a single-time frame,

$$\mathrm{vertex}_{j,k,l,i} = \mathcal{LUMI}_{j,k,l,i}^{\mathrm{exp}}\, \sigma_{\mathrm{production}}\,. \tag{12}$$

3. Comparison with analytical formula and standard numerical integrator

Comparison of ILUMI4d with the analytical formula (no hourglass effect) was done for the following beam parameter settings: $E = 6800$ GeV, particle mass $= 0.938$ GeV, $\epsilon_{n,x,y} = 2.3 \times 10^{-6}$ m, $\beta^*_{x,y} = 0.3$ m, $\sigma_z = 1.2$ ns $(4\,\sigma)$, 1.4×10^{11} nuclei, the crossing angle is changed incrementally from 0 to 400 μrad. The single-bunch luminosity is computed with ILUMI4d and compared to the analytical formula. The change of luminosity as a function of the crossing angle (Fig. 3, upper right) follows the expected formula (13). In Fig. 3, lower left, the longitudinal beam-spot size is compared with the

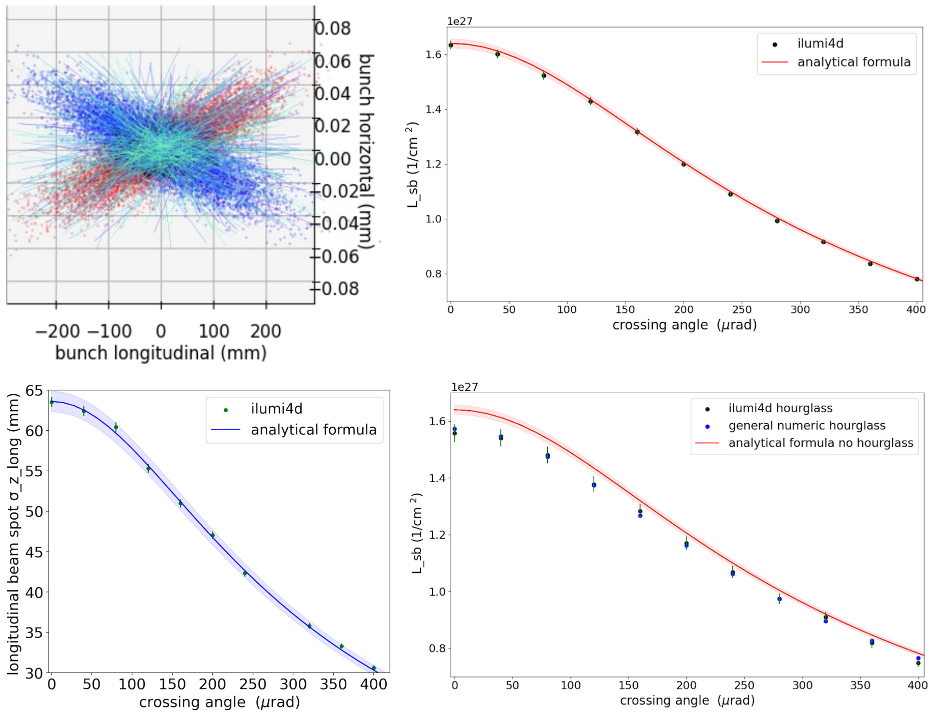

Fig. 3. Upper left: ILUMI4d bunch crossing at $i = 0$ with tracks; upper right: lumi as a function of the crossing angle compared with analytical formula. Lower left: longitudinal bunch size as a function of the crossing angle; lower right: lumi as a function of the crossing angle with the hourglass effect and compared with a numerical integrator.

analytical formula, also following the geometric reduction predicted by the analytical formula [2]. The differences are all smaller than 1.2%. When the hourglass effect is taken into account, the single-bunch luminosity is described by the analytical formula (14) (see [2]) that can only be solved numerically. In Fig. 3, lower right, the lumi reduction as a function of the crossing angle is computed for the same beam parameters as above, but with the hourglass effect taken into account and compared with the standard numerical integrator [4],

$$L_{\mathrm{sb}} = \frac{N_1 N_2}{4\pi\sigma_x\sigma_y}\frac{1}{\sqrt{1+\left(\frac{\sigma_z}{\sigma_x}\frac{\phi}{2}\right)^2}}\,, \tag{13}$$

$$L_{\mathrm{sb,hg}} = \left(\frac{N_1 N_2}{4\pi\sigma_x^*\sigma_y^*}\right)\frac{\cos\frac{\phi}{2}}{\sqrt{\pi}\sigma_z}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{e}^{-z^2 A}}{1+\left(\frac{z}{\beta^*}\right)^2}\mathrm{d}z\,, \tag{14}$$

with

$$\sigma_{x,y}(z) = \sqrt{\beta^* \left(1 + \left(\frac{z}{\beta^*}\right)^2\right) \epsilon_{x,y}}\,,$$

$$A = \frac{\sin^2\frac{\phi}{2}}{(\sigma_x)^2} + \frac{\cos^2\frac{\phi}{2}}{(\sigma_z)^2} = \frac{\sigma_z^2 \sin^2\frac{\phi}{2} + (\sigma_x^*)^2\left[1 + \left(\frac{z}{\beta^*}\right)^2\right]\cos^2\frac{\phi}{2}}{(\sigma_x^*)^2\left[1 + \left(\frac{z}{\beta^*}\right)^2\right]\sigma_z^2}\,.$$

4. Space-time structure of luminosity with crab cavity

Crab cavities [9, 10], installed at both sides of the interaction points, rotate the bunches by the crab angle $\frac{\theta}{2}$ to partially compensate the crossing

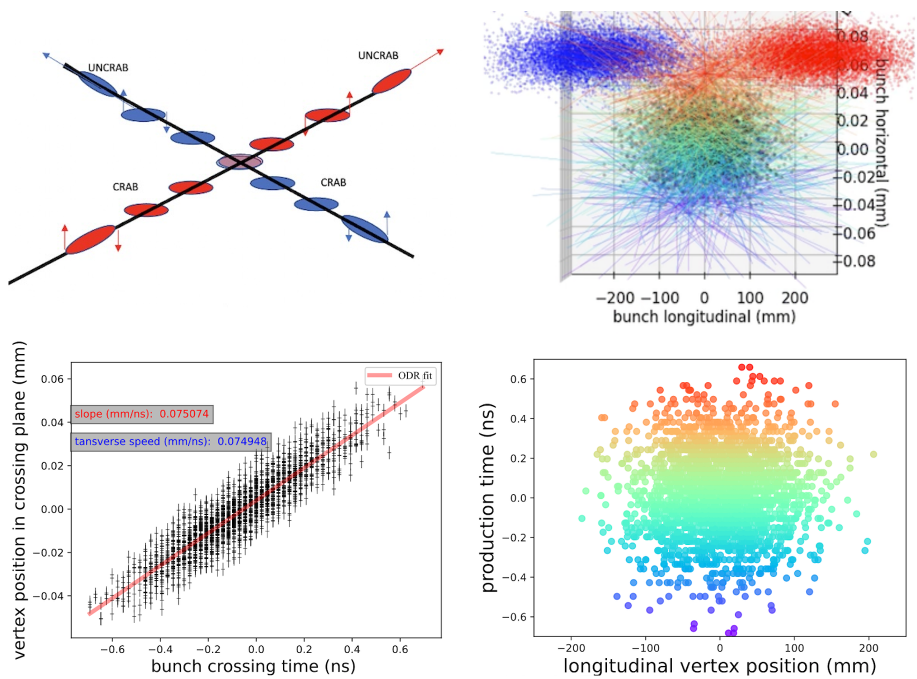

Fig. 4. Upper left: schematic of crab cavity and rotation of bunches; upper right: vertex footprints and tracks for the ideal crab cavity (non-deformed bunches, perfect rotation) compensating half of the full crossing angle 500 μrad. Lower left: vertices generated as a function of the bunch crossing time in the crossing plane at the experiment for non-deformed bunches; lower right: vertex coordinates in the longitudinal direction of the experiment as a function of the crossing time; the colour code of tracks from blue to red: blue = start of collision, red = end of collision.

angle Fig. 4, left. For an ideal (non-deformed) Gaussian bunch pair and perfect rotation, the transverse vertex positions as a function of the production time are computed and fitted with a linear ODR fit [11]. The slope corresponds to the transverse movement induced by the bunch rotation with 0.16% deviation from the expected value ($c\sin(\frac{\theta}{2})$). The vertex distribution in the longitudinal direction as a function of the bunch crossing time is shown in Fig. 4, bottom right.

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