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ENTANGLEMENT ENTROPY, KRYLOV COMPLEXITY, AND DIS DATA*

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I discuss recent results that confirm that one can associate entropy with partonic content of the proton. Furthermore, I show that complexity of the proton is characterized by the parton momentum density function.

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1. Entanglement entropy

Recently there is a growing interest in studies of entropy associated with the system of gluons [1–20]. Here we are interested in the proposal made in [21] where the hadronic entropy is conjectured to be equal to entanglement entropy resulting from bipartition introduced by a virtual photon that resolves partonic system of protons. The conjecture has been shown to be supported by comparison of the entropy associated with the system of gluons to the entropy measured by H1 Collaboration [22–27]. Furthermore, more recently, by crossing symmetry, it has been shown that one can associate hadronic entropy with entropy of fragmentation function [28].

To obtain entropy of partons one can use the Mueller dipole picture [29] leading to the equation that describes the rapidity evolution of probability for the n parton-dipoles state $p_n(y)$. After solving it and evaluating von Neuman entropy, one obtains

$$S(y) = \ln\left(\mathrm{e}^{\lambda y} - 1\right) + \mathrm{e}^{\lambda y} \ln\left(\frac{1}{1-\mathrm{e}^{-\lambda y}}\right) . \tag{1}$$

The expression upon introducing the mean number of dipoles is

$$\bar{n} = \sum_n n p_n \equiv x g(x) , \tag{2}$$

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where $xg(x)$ is the momentum density of gluons and $x = x_0 \mathrm{e}^{-y}$ is the longitudinal momentum fraction carried by the gluon. This can be rewritten as

$$S(\bar{n}) = \ln \bar{n} - (\bar{n} - 1) \ln \left(1 - \frac{1}{\bar{n}}\right) . \tag{3}$$

The gluon density in a high-energy limit has characteristic power-like behavior and in the considered model, it can be approximated as

$$xg(x, \mu) = C_0 \left(\frac{1}{x}\right)^{\Delta(\mu)} . \tag{4}$$

In the above, the hard scale was introduced in order to make the model to be phenomenologically applicable. In order to verify the hypothesis of entanglement entropy associated with the partonic content of proton, the H1 Collaboration performed measurement of hadronic entropy. Two kinds of measurements were made:

— the first scenario addressed measurement in the fixed rapidity window $\Delta\eta = 4$. In this case all the emitted hadrons contribute to the total entropy,

— the second scenario addressed measurement in the moving rapidity window $\Delta\eta = 1.4$. In this case entropy is obtained from counting hadrons in a limited region as specified above.

While the entropy measured in the fixed rapidity window has been successfully described in [22] the description of the moving rapidity window requires generalization of Eq. (3). One has to account for the fact that particles are not measured in a certain region of rapidity. Such generalization leads to [30]

$$S_{\mathrm{loc}} = -\tilde{p}_0 \ln \tilde{p}_0 - (1 - \tilde{p}_0) \ln (1 - p_0) + (1 - \tilde{p}_0)\, S(\bar{n}) , \tag{5}$$

where $p_0(y, y_0) = C_0 \mathrm{e}^{-\Delta(y - y_0)}$ and $y - y_0$ is interval where the particles are measured. In Fig. 1, the results for the description of entropy in fixed and moving rapidity windows are shown. The theory curves were obtained using $C_0 = 0.7$ as follows from the fit of the model of gluon density to the HERA parton density function.

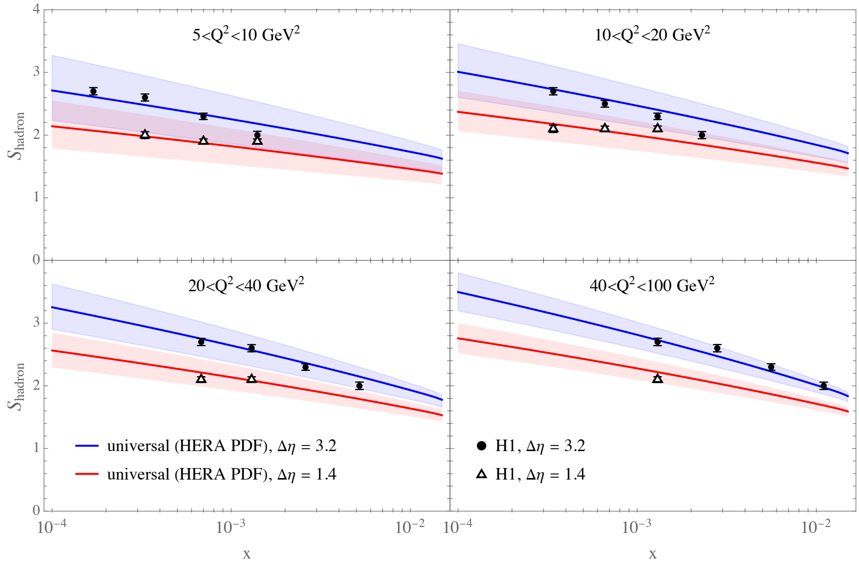

Fig. 1. Red line — entanglement entropy calculated for the moving rapidity window scenario. Blue line — entanglement entropy calculated for the fixed rapidity window scenario.

2. Quantum measures

The entanglement entropy of the parton density is one of the quantum measures. One can however think about other measures of entanglement characterizing system of gluons. Those are capacity of entanglement, complexity, variance, and purity. In the recent study [31], the dipole model discussed above has been shown to map onto an equation for probabilities that follows from Schrödinger equation with the boost-type operator with SL(2,R) symmetry. In particular, it is has been shown that the Krylov complexity $C_{\rm K}$ of such a quantum system corresponds to momentum density $xg(x)$ of gluons *i.e.*

$$C_{\rm K} = xg(x)\,. \tag{6}$$

This result means that gluon density in the low-x regime characterizes complexity of proton. In Fig. 2, we plot various quantum measures. The capacity of entanglement saturates at 1 showing that the system is maximally mixed, this can be also seen by examining the purity which drops to zero. Complexity grows exponentially fulfilling the bound on complexity [32].

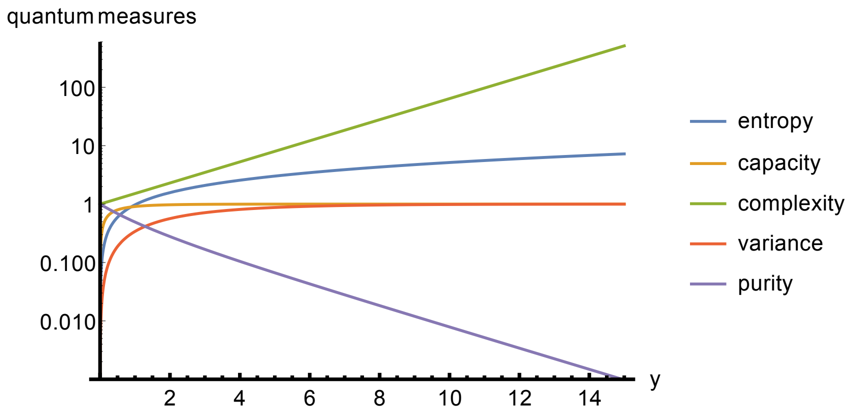

Fig. 2. Quantum measures applied as follow from the dipole model.

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