QUARKONIA IN NON-COMMUTATIVE SPACE*

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Quarkonium bound states are especially promising candidates to test the probable quantum structure of space-time since they represent a system with a reasonably small characteristic distance. In this contribution, we insert this system in a 3-dimensional rotationally-invariant space which is composed of concentric fuzzy spheres of increasing radius called the fuzzy onion. Our aim is to extract some consequences of the space's non-trivial structure on the quarkonia's properties.

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1. Introduction

Considering small length scales, approximately of the order of the Planck's length, a discrete quantum structure of the space is expected to emerge. One of the methods for describing this structure stems from non-commutative (NC) spaces [1]. Opposing to the usual manifolds, coordinates in such spaces are described by non-commuting operators. In the formulation of quantum mechanics on the NC spaces, we can define the NC versions of all the relevant operators and then look for the solutions of the corresponding Schrödinger equation [2].

The space is constructed as an infinite set of fuzzy spheres [1] with growing radius, which form a layered structure one could call a fuzzy onion. Our choice is the two quarks [3], which orbit so close to each other that they probe any new features of the space-time better than larger systems [2].

In Section 2, the classical space is tackled: we introduce the Cornell potential, describe the radial WKB approximation, and apply it for the quarkonia. In Section 3, the NC space is tackled: we formulate the mathematical description of the NC QM, show the exact solution for the NC hydrogen atom, and determine the NC modifications to the quarkonium mass spectra.

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2. Quarkonia in the standard QM

We will deal with the mass spectrum of the bound states of two heavy non-relativistic quarks with masses m_1 and m_2, where the interaction is described by the Cornell potential

$$V_Q(r) = -\frac{C}{r} + B\,r\,. \tag{1}$$

The linear part is responsible for the quark confinement and the r^{-1} part describes the electrostatic-like interaction between charged quarks. Both B and C will be treated as free parameters and will be fixed by experimental data. The observed mass of the bound state is then

$$M_{nl} = m_1 + m_2 + E_{nl}\,. \tag{2}$$

On the one hand, we will look for corrections due to the NC structure of the space, but we will also present a modified way to treat the QM problem with the Cornell potential in the classical space. The binding energies E_{nl} can be determined from the WKB condition

$$\frac{1}{\hbar}\int_{r_1}^{r_2} \mathrm{d}r\, p(r) = \left(n + \frac{1}{2}\right)\pi\,, \qquad n \in \mathbb{Z}_0^+\,, \tag{3}$$

where $p(r)$ is the particle's semi-classical momentum and $p(r_1) = p(r_2) = 0$.

To solve the integral (3) for the Cornell potential (1), we use the Pekeris-type approximation [4], which describes the expansion of $p(r)$ around the typical distance of the quarkonia $r_Q = \sqrt{C/B}$. After the calculations [5], we get the binding energy of the quarkonia in the commutative classical space

$$E_{nl} = -\frac{2\mu}{\hbar^2}\left[\frac{2C}{n + \frac{1}{2} + \sqrt{\frac{2\mu}{\hbar^2}C\sqrt{\frac{C}{B}} + \left(l + \frac{1}{2}\right)^2}}\right]^2 + 3\sqrt{BC}\,, \tag{4}$$

where μ is the reduced mass. We substituted the experimental data for $1S$ and $2S$ states [3] into equation (4), hence we obtained the free parameters C and B by solving the two equations numerically for the given system. Let us point out that our results for the masses (gathered in Table 1) are in good agreement with the experimental data and stand their ground among other, arguably more sophisticated models, *e.g.* [6].

The data for the bottomonium and the bottom-charmed meson states were collected to analogous tables [5]. From them, we have obtained the typical distance $r_Q \approx 10^{-16}$ m, which is known to be roughly the size of the quark bound states [3].

Table 1. Mass spectrum of the $c\bar{c}$ meson.

$c\bar{c}$ meson	$\mu = 2.54$ GeV	$B = 0.322$ GeV^2	$C = 0.891$
State	Particle	Present work M_{nl} [GeV]	Exp. data M_{nl} [GeV]
$1S$	$J/\psi(1S)$	used for B, C	3.097
$2S$	$\psi(2S)$	used for B, C	3.686
$3S$	$\psi(4040)$	3.889	4.040
$1P$	$\chi_{C1}(1P)$	3.518	3.511
$2P$	$\chi_{C2}(3930)$	3.823	3.923
$1D$	$\psi(3770)$	3.787	3.774

3. Quarkonia in the NC QM

The NC space $\mathbb{R}^3_\lambda$ can be described by a model of concentric fuzzy spheres with increasing radius. The commutator of coordinates is defined as

$$[x_i, x_j] = 2\,i\lambda\varepsilon_{ijk}x_k\,, \tag{5}$$

where the parameter λ describes the fuzziness of the space structure. An auxiliary Fock space is introduced accompanied by two sets of creation $a^\dagger$ and the annihilation a operators; their commutation relations are

$$\left[a_\alpha, a^\dagger_\beta\right] = \delta_{\alpha\beta}\,, \qquad \left[a_\alpha, a_\beta\right] = \left[a^\dagger_\alpha, a^\dagger_\beta\right] = 0\,, \qquad \alpha,\beta \in \{1,2\}\,. \tag{6}$$

The great news is that all the operators of all the relevant physical quantities and the wave function Ψ can be constructed with this choice of the $a^\dagger$ and a operators. The definition of the position operator obeys the commutator (5)

$$x_j = \lambda\sigma^j_{\alpha\beta}a^\dagger_\alpha a_\beta\,, \qquad j \in \{1,2,3\}\,, \tag{7}$$

where σ^j is the corresponding Pauli matrix. The Hamiltonian operator becomes

$$\hat{H}_\lambda\Psi = \frac{\hbar^2}{2\lambda\mu r}\left[\hat{a}^\dagger_\alpha, [\hat{a}_\alpha, \Psi]\right] + V(\hat{r})\Psi\,. \tag{8}$$

Both the exact and the WKB solution of the Schrödinger equation with (8) for the Coulomb potential $V_\mathrm{C}(r) = -e^2/(4\pi\varepsilon_0 r)$ in $\mathbb{R}^3_\lambda$ [2] give

$$E^\lambda_N = \frac{\hbar^2}{m_e\lambda^2}\left(1 - \sqrt{1 + \frac{m_e^2c^2\alpha^2}{\hbar^2}\frac{\lambda^2}{N^2}}\right)\,. \tag{9}$$

The WKB solution of the Schrödinger equation for the Cornell potential $V_Q(r)$ with the Pekeris-type approximation in $\mathbb{R}^3_\lambda$ [5], where $c = 2\mu C r_Q/\hbar^2$ and $b = 2\mu B r_Q^3/\hbar^2$ are the dimensionless parameters of the potential, is

$$E_{nl}^{\sigma} = E_{nl} + \sigma^2 \frac{\hbar^2}{2\mu}\frac{B}{C}\left(\frac{b\left(105b^2+62bc+9c^2\right)+4b(c+3b)l(l+1)}{8\left[n+\frac{1}{2}+\sqrt{b+\left(l+\frac{1}{2}\right)^2}\right]^2}\right.$$
$$-\frac{b}{4}(15b+4c)+\frac{b(c+3b)^4}{8\sqrt{b+\left(l+\frac{1}{2}\right)^2}\left[n+\frac{1}{2}+\sqrt{b+\left(l+\frac{1}{2}\right)^2}\right]^5}$$
$$\left.-\frac{(45b-c)(c+3b)^3}{64\left[n+\frac{1}{2}+\sqrt{b+\left(l+\frac{1}{2}\right)^2}\right]^4}\right)+\mathcal{O}\left(\sigma^2\right), \qquad (10)$$

where $\sigma = \lambda/r_Q$ is the dimensionless fundamental length scale. Calculated from the Planck's length ($\lambda \approx 10^{-35}$ m), the potential effect of the non-commutativity of the space is on the 39^{th} decimal place in the mass spectrum of the mesons [5]. These modifications are beyond the limit of accuracy of any current measurement. However, we can tackle a reversed problem where we use the uncertainty of the mass of the most precisely measured meson, the $J/\psi(1S)$ particle — $M_{00} = (3096.900 \pm 0.006)$ MeV, to give an upper bound for the extent of fuzziness: $\lambda \leq 1.11 \times 10^{-18}$ m.

4. Conclusions

We have described the derivation of the masses of the considered mesons and the first non-trivial correction due to the NC structure, which is of the order of 10^{-39}, which means we are still far from any reasonably measurable contribution. In the future, it would be interesting to look further for other systems, where the effect of the NC space-time could be detected.

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