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MODIFIED BLACK HOLE WITH EXTRA DIMENSIONS AS AN UNUSUAL DARK MATTER CANDIDATE*

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By considering an ansatz for $(1 + (3 + n))$-dimensional static spacetime with three-dimensional spherical symmetry, we find different classes of vacuum solutions of Einstein field equations. A class of solutions with nontrivial extension of the Schwarzschild spacetime with extra dimensions features unusual properties, which may provide a possibility to address problems of dark matter and dark energy.

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1. Introduction

The Schwarzschild spacetime [1] is one of the most important solutions of Einstein field equations. Its modification to an arbitrary number of space dimensions was first found by Tangherlini [2]. Later, the motivation for extra dimensions originated mainly from Kaluza–Klein models [3] and string theory [4]. This paper provides an analysis of a special class of vacuum solutions of Einstein field equations in $1+(3+n)$ dimensions under the assumption of spherical symmetry in the three-dimensional part and Euclidean symmetry in the n-dimensional part, $\mathrm{SO}(3) \times \mathrm{E}(n)$.

The following Section 2 reveals two sets of such solutions, Section 3 contains an analysis of their properties, and in the final Section 4, we briefly discuss the possibility of potential implications for problems of dark matter and dark energy. We use conventions with the light speed set to unity and $-,+,+,\ldots$ signature. Lowercase Latin indices indicate three space coordinates, $i = 1, 2, 3$, capital Latin indices run through n extra dimensions, $A = 1, \ldots, n$, and all spacetime indices are collectively denoted by Greek letters, $\mu = 0, i, A$.

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2. Vacuum solutions

It is straightforward to find a vacuum solution of Einstein field equations with $\mathrm{SO}(3)\times\mathrm{E}(n)$ symmetry in the form

$$\mathrm{d}s^2 = -f^{\alpha}\mathrm{d}t^2 + f^{\beta}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(2)} + f^{\gamma}\delta_{AB}\mathrm{d}\zeta^A\mathrm{d}\zeta^B\,, \tag{1}$$

where $\mathrm{d}\Omega^2_{(2)} = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$, function f depends only on the radial coordinate r, and α, β, and γ are constants.

There are two sets of solutions different from $(1+(3+n))$-dimensional Minkowski spacetime. For the first one, we have $\gamma = 0$, $\alpha = -\beta$, and the spacetime metric reads

$$\mathrm{d}s^2 = -\left(1+\frac{a}{r}\right)\mathrm{d}t^2 + \left(1+\frac{a}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(2)} + \delta_{AB}\mathrm{d}\zeta^A\mathrm{d}\zeta^B\,. \tag{2}$$

This is only a trivial extension of the original (1+3)-dimensional Schwarzschild solution. The second set of solutions corresponds to $\alpha/\beta = (n-1)/(n+1)$ and $\gamma/\beta = -2/(n+1)$ with the spacetime metric of the form

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1+\frac{a}{r}\right)^{-\frac{n-1}{n+1}}\mathrm{d}t^2 + \left(1+\frac{a}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{(2)} \\ &+\left(1+\frac{a}{r}\right)^{\frac{2}{n+1}}\delta_{AB}\mathrm{d}\zeta^A\mathrm{d}\zeta^B\,.\end{aligned} \tag{3}$$

This is a nontrivial extension of the Schwarzschild solution, and its properties are considerably different from the trivial one.

The metric (3) can also be written in the generalized Weyl form [5]

$$\mathrm{d}s^2 = -e^{2A}\mathrm{d}t^2 + e^{2B}\mathrm{d}\varphi^2 + e^{2C}\delta_{AB}\mathrm{d}\zeta^A\mathrm{d}\zeta^B + e^{2D}\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right), \tag{4}$$

where functions A, B, C, and D depend on ρ and z, and they are solutions of Laplace's equation in cylindrical coordinates (ρ,φ,z) sourced by infinitely thin rod segments. By choosing

$$\begin{aligned} A &= \frac{1}{2}\frac{1-n}{1+n}\ln\frac{L+a}{L-a}\,, &\qquad B &= \ln\rho - \frac{1}{2}\ln\frac{L+a}{L-a}\,, \\ C &= \frac{1}{1+n}\ln\frac{L+a}{L-a}\,, & D &= \frac{1}{2}\ln\frac{(L-a)^2}{4R_+R_-}\,, \end{aligned} \tag{5}$$

where $L = R_+ + R_-$, $R_\pm = \sqrt{(z\pm a/2)^2+\rho^2}$, and by performing the coordinate transformation $z = r(1+a/(2r))\cos\vartheta$, $\rho = r\sqrt{1+a/r}\sin\vartheta$, we obtain the metric (3).

3. Physical properties

Two classes of solutions, (2) and (3), differ from each other in terms of their physical properties. We assume that extra dimensions are compactified into a microscopic flat n-torus with $S^1 \times \ldots \times S^1$ topology, so that the studied spacetimes are applicable to our $(1+3)$-dimensional Universe. Macroscopic objects can then move only in three space dimensions.

Singularities can be found by calculating the Kretschmann scalar defined as $K = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. For the trivial extension (2), we have $K = 12ar^{-6}$, and therefore, there is a physical singularity only at the center. On the other hand, for the nontrivial extension (3), the Kretschmann scalar is more complicated,

$$K = \frac{12a^2}{r^6}\left[1 - \frac{1}{3}\frac{n(n-1)}{(n+1)^3}\left(1+\frac{r}{a}\right)^{-2}\left(\frac{3}{4}n + 1 + (n+1)\frac{r}{a}\right)\right]. \qquad (6)$$

This indicates a physical horizon not only at the center but also a shell singularity on the horizon, $r = -a$, in the case with negative a and $n > 1$. Solutions of this type are called Kaluza–Klein bubbles, first found in [6].

Another important difference follows from fixing the constant a by taking the Newtonian limit. A small test object moving very slowly in $(1+3)$ dimensions behaves as a test particle in a Newtonian gravitational field generated by a point mass M. In the case with the trivial extension (2), the relation between a and M is $a = -2GM$, where G is the Newtonian gravitational constant, the same as for the original Schwarzschild solution. In the nontrivial case (3), it is $a = 2GM(n+1)/(n-1)$, which implies positive a for positive M, if $n > 1$, and absence of horizon, *i.e.*, we have naked singularities. For $n = 1$, the value of a cannot be fixed, and $M = 0$.

Finally, we can evaluate a conserved quantity associated with the found spacetimes. The conserved energy $\mathcal{E}$ can be defined through the Landau–Lifshitz stress-energy pseudotensor [7] as

$$\mathcal{E} = \oint_{\partial\Omega} h^{00\mu}\mathrm{d}S_\mu\,, \qquad h^{\mu\nu\lambda} = \frac{1}{16\pi\kappa}\left[(-g)\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma}\right)\right]_{,\sigma}. \qquad (7)$$

By choosing the $(2+n)$-dimensional boundary $\partial\Omega$ to correspond to a sphere in three dimensions, using an isotropic radial coordinate R defined by the transformation $r = (1 - a(4R))^2 R$, $R \in [1/4, \infty)$, for $a < 0$ covering only the region above the horizon, and taking the limit of the radius of the sphere going to infinity, we find $\mathcal{E} = M$ in the trivial case (2), while in the nontrivial case (3), we have $\mathcal{E} = -M/(n-1)$. Therefore, for the nontrivial extension and $n > 1$, the conserved energy defined through the Landau–Lifshitz stress-energy pseudotensor has the opposite sign as the mass obtained from the Newtonian limit.

4. Discussion

We have studied properties of higher-dimensional extensions of the original Schwarzschild spacetime with an arbitrary number of extra dimensions n. There are two classes of such spacetimes, (2) and (3).

For the nontrivial extension of the Schwarzschild solution (3) with the number of extra dimensions $n > 1$, we have a repulsive gravitational force in the Newtonian limit, $M < 0$, with positive conserved mass $\mathcal{E}$. If our Universe contained objects with such properties, they would contribute to overall energy density while not participating in the formation of the nonhomogeneous cosmic structure. Then they could explain not all, but at least a part of the dark matter. Unfortunately, their peculiar properties suggest nonphysicality and potential problems with stability.

However, the fundamental structure of the spacetime may allow for the existence of microscopic and short-lived regions within so-called quantum foam with exotic properties corresponding to the nontrivial extension of the Schwarzschild spacetime (3). This may have implications for the dark energy problem in cosmology. We are leaving this open question for future work.

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