



HADRONIC FORM FACTORS IN QCD AND THE INCOMPLETENESS PROBLEM IN THE TIME-LIKE REGION* **

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Hadronic form factors fulfill dispersion relations and superconvergence sum rules for their spectral density as genuine imprints of QCD. We show several instances where these conditions are flagrantly violated due to the lack of information in the region above the largest known resonance mass and below the onset of perturbative QCD. We propose to use radial Regge trajectories to fill this gap and examine the consequences of such a “minimal” spectral hadronic ansatz. We illustrate the results with the pion charge form factor.

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1. Introduction

Hadronic form factors (FFs) have been around since Hofstadter proved the finite proton size in 1955 from the electron scattering experiments; they are identified as matrix elements of the electromagnetic current. Since then, the nucleon and other hadronic FFs have been measured or computed in lattice QCD using all sorts of currents: electromagnetic, axial, gravitational,

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etc. Analyticity, crossing, chiral symmetry, and pQCD play a key role here and imply a set of normalization and superconvergence sum rules (SCSRs), which are rigorous conditions. Besides, hadronic FFs provide an access to the distributions of matter, charge, currents or stresses inside a hadron. When the currents are conserved, FFs satisfy normalization conditions. On the other hand, confinement requires hadronic physical states to be color singlet, but what is the complete set of eigenstates of QCD spanning the physical Hilbert space, \mathcal{H}_{QCD} ? Moreover, how can we characterize this completeness property in practice? In this paper, we are concerned with the idea of completeness in hadronic matrix elements and their associated FFs, defined such that their spectral decomposition has definite $J^{PC}IG$ quantum numbers.

2. Analyticity and sum rules

Schematically, to simplify the presentation¹, we define

$$\langle H(p') | J | H(p) \rangle = F(q^2), \quad q = p' - p, \quad (1)$$

where $F(t)$ is the FF and $|H(p)\rangle$ is the hadronic state. For conserved currents, $F(0)$ is usually known from symmetry requirements and the associated Ward–Takahashi identities. Due to crossing, $F(t)$ is analytic in the complex t -plane, except for branch cuts along the positive real axis above $t > s_{\text{th}}$, corresponding to the lowest threshold production with suitable quantum numbers. QCD implies a set of rigorous sum rules based on the asymptotic behavior in the deep space-like region $q^2 = -Q^2 \rightarrow \infty$, characterized by the running coupling constant and its analytical extrapolation to the complex plane $q^2 = |s| e^{i \arg(s)}$, with $0 \leq \arg(s) < 2\pi$

$$\alpha(s) = \frac{4\pi/\beta_0}{\log(|s|/\Lambda_{\text{QCD}}^2) + i(\arg s - \pi)} \Rightarrow \alpha(-Q^2) = \frac{4\pi}{\beta_0} \frac{1}{\log(Q^2/\Lambda_{\text{QCD}}^2)}. \quad (2)$$

For most cases, one- or two-gluon exchange implies the asymptotics [2, 3]

$$F(s) \sim \begin{cases} \frac{\alpha(s)}{s} \sim \frac{1}{s(\log s)} & \text{mesons} \\ \frac{\alpha(s)^2}{s^2} \sim \frac{1}{s^2(\log s)^2} & \text{baryons} \end{cases}. \quad (3)$$

¹ For a more comprehensive description of the pion and nucleon FFs in terms of the Lorentz invariant operators, see, *e.g.*, [1] and references therein.

Consequently, $\lim_{|s| \rightarrow \infty} F(s) = 0$ ². The point $s = s_{\text{th}}$ marks the origin of a branch cut, which goes up to $s \rightarrow \infty \pm i\epsilon$ with a discontinuity $\text{Disc}F(s) = 2i\text{Im}F(s)$. Cauchy's theorem applied to a closed contour encircling the branch cut, yields an *unsubtracted* dispersion relation (DR) of the form

$$F(t) = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\text{Im} F(s)}{s-t} \implies F(0) = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds \frac{\text{Im} F(s)}{s}. \quad (4)$$

This sum rule implies that the knowledge of the process $J \rightarrow H\bar{H}$ from the lowest (possibly unphysical) threshold up to infinity allows one to compute $F(0)$ directly³. In addition, we also have the SCSRs

$$0 = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds \text{Im} F(s) \quad \text{mesons and baryons}, \quad (5)$$

$$0 = \frac{1}{\pi} \int_{s_{\text{th}}}^{\infty} ds s \text{Im} F(s) \quad \text{baryons}, \quad (6)$$

simply following from the extra suppression with powers of s in Eq. (3). The normalization and SCSR are rigorous theorems in QCD in the time-like region and direct consequences of the completeness of hadronic states. These properties imply that the spectral functions *cannot* be positive-definite in the whole domain. Still, for the elastic single-channel case, Watson's theorem implies, $\text{Im} F(s) = |F(s)| \sin \delta$ for $s_{\text{th}} \leq s \leq s_{\text{in}}$, where s_{in} is the threshold for the first inelastic channel and δ denotes the phase-shift for the elastic $\bar{H}H \rightarrow \bar{H}H$ scattering process. For an attractive interaction $\delta > 0$ and in this case, $\text{Im} F(s) > 0$ below s_{in} . This happens in particular for a resonating channel, $\bar{H}H \rightarrow R \rightarrow \bar{H}H$.

3. Violations of the sum rules in practice

The workings of SCSRs are best appreciated in the case of the pion charge FF, which has been analyzed in the time-like region up to $\sqrt{s_{\text{max}}} = 3$ GeV in terms of a phase-modulus DR [6] with $F_Q(0) = 1$. From there, one can

² These are leading order contributions, whose proportionality constants are, in general, scale- and scheme-dependent. Higher-order contributions are not well converging for the currently available range in measurements. Moreover, the onset of pQCD may happen at extremely large momenta (see, *e.g.*, Refs. [4, 5] for the pion charge FF.)

³ This is a *prediction* when normalization is not protected by symmetry. Prominent cases are the magnetic moments or the Druck-term, $D(0)$, of the nucleon.

extract $\text{Im } F_Q^\pi(s)$ in the range $4m_\pi^2 \leq s \leq s_{\max}$ [7, 8], with $\text{Im } F_Q^\pi(s) > 0$ for $4m_\pi^2 \leq s \leq s_{\text{in}} = 4m_K^2$, to find

$$\frac{1}{\pi} \int_{s_{\text{th}}}^{s_{\max}} ds \frac{\text{Im } F_Q^\pi(s)}{s} \sim 1, \quad \frac{1}{\pi} \int_{s_{\text{th}}}^{s_{\max}} ds \text{Im } F_Q^\pi(s) \sim m_\rho^2. \quad (7)$$

On the other hand, in pQCD to LO, one obtains for $s \rightarrow \infty$ [2]

$$\frac{1}{\pi} \text{Im } F_Q^\pi(s) \rightarrow -\frac{16\pi F_\pi^2}{s} \text{Im} \alpha_s = -\frac{64\pi^2 F_\pi^2}{\beta_0 s} \frac{1}{\log^2(s/\Lambda_{\text{QCD}}^2) + \pi^2} < 0, \quad (8)$$

since $\text{Im} \alpha_s > 0$ (*cf.* Ref [9] and Eq. (2)). Numerically, the contributions to these sum rules from pQCD are only -0.0025 and -0.114 GeV^2 , respectively, even in the unrealistic scenario, where pQCD extends down to $s = s_{\max}$ [7, 8]. The situation is similar when FFs are deduced from involved Roy–Steiner analyses involving the coupled $\pi\pi$ and $K\bar{K}$ channels, and implementing unitarity and crossing *below* the $N\bar{N}$ threshold, which acts as a high-energy cut-off, $\Lambda = 2m_N$. For instance, the contributions to SCSRs for the π and K gravitational FFs [10], where $s_{\text{th}} = s_\pi = 4m_\pi^2$, is

$$\frac{1}{\pi} \int_{s_\pi}^{\infty} ds \text{Im } A^{\pi,K}(s) = 0, \quad \begin{cases} \frac{1}{\pi} \int_{s_\pi}^{4m_N^2} ds \text{Im } A^\pi(s) = 1.86 \text{ GeV}^2 \\ \frac{1}{\pi} \int_{s_\pi}^{4m_N^2} ds \text{Im } A^K(s) = 0.77 \text{ GeV}^2 \end{cases}. \quad (9)$$

For the nucleon, strong violations occur already for the normalization ($n = 0$). This is illustrated in Table 1, which summarizes a collection of sum rules for the nucleon isovector–vector electric and magnetic FFs [11], as well as the gravitational FFs [10]. They correspond to the integrals

$$S_n(\Lambda) \equiv \frac{1}{\pi} \int_{s_\pi}^{\Lambda^2} ds \frac{\text{Im } F(s)}{s} s^n, \quad n = 0, 1, 2. \quad (10)$$

The flagrant violations are due to the finiteness of the upper cut-off $\Lambda = 2m_N$ and have been mended by introducing effective averaging of narrow resonances, unrelated to the PDG ones, designed to fulfill all sum rules [10, 11].

Table 1. Sum rules from the Roy–Steiner analyses for the nucleon [10, 11]. A dash indicates that the value is not provided.

	$n = 0$		$n = 1$		$n = 2$	
A	$2m_N$	∞	$2m_N$	∞	$2m_N$	∞
G_E^v	0.68(11)	1/2	—	0	—	0
G_M^v	3.21(30)	2.35	—	0	—	0
A	1.8	1	2.6	0	4.0	0
$2J$	1.4	1	1.9	0	2.8	0
Θ/m_N	2.6	1	3.7	0	8.1	0

4. Minimal spectral hadronic ansatz

An important application of the DR is to provide the FFs in the space-like region, where the time-like region details become rather irrelevant. In this state of affairs, the old meson dominance approach becomes rather simple and reasonably accurate. From the QCD viewpoint, it is based on the narrow character of hadronic resonances, an assumption motivated by the large- N_c limit. There, the width to mass ratio is $\Gamma/M = \mathcal{O}(N_c^{-1}) \sim 1/3$, which is strongly supported by the experimental Suranyi’s average ratio $\langle \Gamma/M \rangle_{\text{PDG}} = 0.12(18)$ [12]. The spectral function becomes a sum of *infinitely many* narrow resonances

$$\frac{1}{\pi} \text{Im} F(s) = \sum_{i=0}^{\infty} c_i \delta(s - m_i^2) \implies F(t) = \sum_{i=0}^{\infty} c_i \frac{m_i^2}{m_i^2 - t}. \quad (11)$$

The minimal hadronic antatz frequently used in the resonance physics is based on saturating (11) with a *finite* number of narrow resonances (typically one or two), disregarding the slow logarithmic scale running of α_s . The half-width rule can be used as a crude estimate of uncertainties [1, 13, 14]. This truncation only violates the highest SCSR. A clear example is provided by the charge FF of the pion, as given by the meson dominance model, $F(-Q^2) = m_\rho^2/(m_\rho^2 + Q^2)$. Schematically, SCSR becomes

$$0 = \underbrace{\frac{1}{\pi} \int_{s_{\text{th}}}^{s_{\text{max}}} ds \text{Im} F(s)}_{m_\rho^2} + \underbrace{\frac{1}{\pi} \int_{s_{\text{max}}}^{s_{\text{pQCD}}} ds \text{Im} F(s)}_{-m_\rho^2 + \eta} + \underbrace{\frac{1}{\pi} \int_{s_{\text{pQCD}}}^{\infty} ds \text{Im} F(s)}_{-\eta}, \quad (12)$$

where the first and the last term on the r.h.s. represent the meson dominance and the genuinely small pQCD contributions, respectively. Additional

missing states (second term) are necessary to compensate. To fill the gap, a model of the spectral density for $s_{\max} \leq s \leq s_{\text{pQCD}}$, based on the radial Regge trajectories, $m_n^2 = an + b$ [12], was proposed in Ref. [9]. In particular,

$$\frac{1}{\pi} \text{Im} F(s) = \begin{cases} \rho_{\text{ChPT}}(s) & 4m_\pi^2 \leq s \leq 16m_\pi^2, & \text{threshold region} \\ \rho_{\text{R}}(s) & 16m_\pi^2 \leq s \leq \Lambda_{\text{R}}^2, & \text{isolated resonance region} \\ \rho_{\text{Reg}}(s) & \Lambda_{\text{R}}^2 \leq s \leq \Lambda_{\text{pQCD}}^2, & \text{Regge region} \\ \rho_{\text{pQCD}}(s) & \Lambda_{\text{pQCD}}^2 \leq s \leq \infty, & \text{pQCD region} \end{cases}$$

In the Regge region, $s_{\max} = \Lambda_{\text{R}}^2 \leq s \leq \Lambda_{\text{pQCD}}^2$, the finiteness of the overlapping radial resonances with a fixed Suranyi's ratio $\Gamma_n/M_n \sim 0.12$ is compatible with a behavior of the form $1/s^{1+\epsilon}$ for mesons and $1/s^{2+2\epsilon}$ for baryons, which precludes the pQCD $\alpha_s/s^{1,2}$ behavior. The value of ϵ depends on the matching point with pQCD, where the absolute normalization is scale-dependent and not very stable perturbatively. For that reason, we use a logarithmic derivative matching

$$\frac{d}{ds} \log \rho_{\text{Reg}}(s)|_{s=s_{\text{pQCD}}} = \frac{d}{ds} \log \rho_{\text{pQCD}}(s)|_{s=s_{\text{pQCD}}}, \quad (13)$$

which gives $\epsilon \sim 1/\log(s_{\text{pQCD}}/\Lambda_{\text{QCD}}^2)$ asymptotically and yields $\epsilon \sim 0.05$ – 0.1 for a very wide range of scales, see Fig. 1 (left), which can be taken as a systematic error. After all conditions are imposed, the explicit pQCD and threshold pieces become tiny (see, *e.g.*, [8, 9]), and the FF reads

$$F(-Q^2) = \sum_{n=1}^N c_n \frac{m_n^2}{m_n^2 + Q^2} + \frac{\text{Im} F(s_{\text{R}})}{\pi} \int_{s_{\text{R}}}^{\infty} ds \left(\frac{s_{\text{R}}}{s}\right)^{N+N\epsilon} \frac{1}{s + Q^2}, \quad (14)$$

with $N_{\text{mes}} = 1$ and $N_{\text{bar}} = 2$. The sum rules are used to fix c_n and $\text{Im} F(s_{\text{R}})$.

We illustrate the procedure for the charge pion form factor (see also [15] for the gravitational FF). In addition to vector meson dominance with the ρ -meson, we take the Regge contribution from $\sqrt{s_{\text{R}}} = m_{\rho'}$, so that

$$F_Q(t) = \frac{Z_\rho m_\rho^2}{m_\rho^2 - t} + \frac{\text{Im} F_Q(s_{\text{R}})}{\pi} \int_{s_{\text{R}}}^{\infty} ds \left(\frac{s_{\text{R}}}{s}\right)^{1+\epsilon} \frac{1}{s - t}. \quad (15)$$

The normalization and superconvergence conditions follow from

$$1 = Z_\rho + \frac{\text{Im} F_Q(s_{\text{R}})}{\pi(1+\epsilon)}, \quad 0 = Z_\rho m_\rho^2 + \frac{\text{Im} F_Q(s_{\text{R}}) s_{\text{R}}}{\pi\epsilon}, \quad (16)$$

respectively. Eliminating $\text{Im} F_Q(s_{\text{R}})$ and Z_ρ , we see that the Regge contribution vanishes for $\epsilon \rightarrow 0$, which corresponds to taking $s_{\text{pQCD}} \rightarrow \infty$. In Fig. 1 (right), we can see the departure from simple VMD with the mass value $m_\rho = 0.77$ GeV compared with the full FF for $\epsilon = 0.05$ – 0.1 and data.

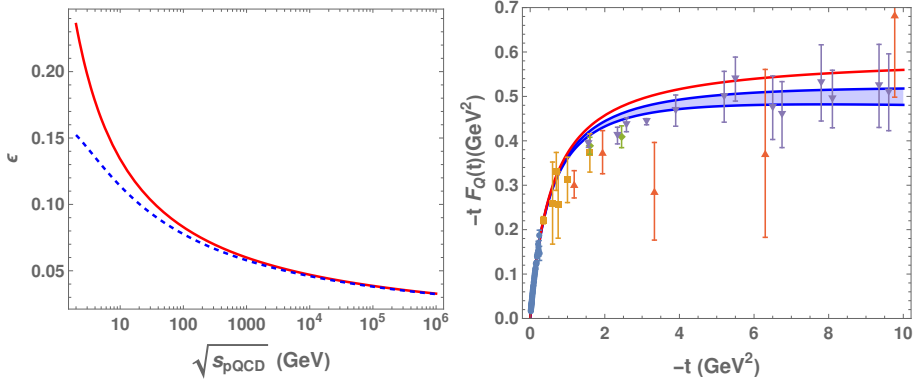


Fig. 1. Left panel: Regge- ϵ as a function of the matching scale with pQCD, $\sqrt{s_{\text{pQCD}}}$; Eq. (13) (dashed) and asymptotic (solid). Right panel: Pion charge form factor from simple VMD with $m_\rho = 0.77$ GeV compared with the FF obtained from Regge matching at the Regge scale $s_R = m_{\rho'}^2$ and for $\epsilon = 0.05\text{--}0.1$ (solid band). For experimental and lattice QCD data, see Refs. [6, 7] and references therein.

5. Conclusions

Dispersive methods have a reputation of being rigorous and useful, until it comes to practical applications, where a high-energy cut-off becomes explicit and influential. For hadronic form factors, most analyses ignore and/or violate the QCD superconvergence sum rules. The infinite radial Regge towers of states provide a simple framework where the smallness of the high-energy tail becomes natural but not negligible. While the precise onset of pQCD is unknown, our scheme provides a way to estimate this systematic uncertainty, which is rather competitive in the studied applications.

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