ENTROPY AND MEAN MULTIPLICITY FROM DIPOLE MODELS IN THE HIGH-ENERGY LIMIT*

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The 1D Mueller dipole model, its high-energy limit and its generalization were investigated. To address the ambiguity stemming from different definitions of the pseudorapidity ranges in experimental measurements, we propose the entropy as a function of the logarithm of the average multiplicity, $S(\ln\langle n\rangle)$, as a universal observable. From the solutions of the models, we calculate both the entropy and the average charged particle multiplicity and compare to data measured in proton–proton collisions. We obtained these quantities directly from the measured charged particle multiplicity distributions and determine the model parameters via fits. We find that the generalized dipole model provides a significantly better description of the data than the 1D Mueller model.

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1. Introduction

Charged particle multiplicity distributions have been measured since the dawn of experimental particle physics. Increasing interest is observed nowadays in such data due to recent theoretical proposal that links the Shannon's entropy that is associated to the multiplicities to the entanglement entropy of the initial partonic system. It is conjectured that the initial state in the high-energy limit is maximally entangled, therefore the associated entanglement entropy is maximal and can be expressed as $S_{\rm init} = \ln\langle n\rangle$, where $\langle n\rangle$

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is the average number of partonic microstates. This conjecture was investigated in deep inelastic scattering experiments with the H1 detector [1] and in recent theory calculations based on parton distribution functions [2–4]. Good agreement was found between the calculations and the data.

To describe the multiplicities in the final state, the maximum entropy principle [5–7] can also be utilized. It was shown, however, that the shape of the distributions is different for the definitions of the pseudorapidity variable η adopted by HERA [1] and LHCb [8], where a narrow $\Delta\eta$ window is shifted across the acceptance, and for those adopted by ALICE [9–11], ATLAS [12, 13], CMS [14], and UA5 [15], where the rapidity window with increasing width is centered at $\eta = 0$. Such a difference requires extra attention when one compares model calculations or measurement results.

Our proposition [16] is to use the universal entropy function $S(\ln\langle n\rangle)$ to compare measured entropies from different experiments and model calculations. This observable is straightforward to determine from the experimental data using the definition

$$\langle n\rangle = \sum_n nP(n) \quad \text{and} \quad S_h = -\sum_n P(n)\ln(P(n))\,. \tag{1}$$

We obtain the universal $S(\ln\langle n\rangle)$ from the charged particle multiplicity distribution measured in $p+p$ collisions [8, 10–15, 17] and compare it to model calculations such as the 1D Mueller dipole model [18–20] and one of its extensions [21].

2. The dipole cascade models

The dipole models used are the 1D Mueller model [18–20] and its generalization [16, 21], which describe parton cascades that, in the 1D case, develops in rapidity. The generalized model has an additional parameter that allows for more accurate description of multiplicity.

2.1. 1D Mueller model

The 1D Mueller model that depends only on rapidity is given by the following cascade equation:

$$\partial_y P_n(y) = -\alpha n P_n(y) + (n-1)\alpha P_{n-1}(y)\,, \tag{2}$$

where $P_n(y)$ is the initial-state parton multiplicity distribution and α represents the gluon emission kernel and characterizes the speed of the Balitsky–Fadin–Kuraev–Lipatov (BFKL) cascade evolution [22, 23]. The solution of the equation can be written in the form of

$$P_n(y) = \frac{1}{C}\,\mathrm{e}^{-\alpha y}\left(1 - \frac{1}{C}\,\mathrm{e}^{-\alpha y}\right)^{n-1}\,, \tag{3}$$

with the recently introduced C parameter [3]. One can observe that, by introducing the probability $p = \frac{1}{C}\,\mathrm{e}^{-\alpha y}$, the solution has the form of the geometric distribution, therefore the mean multiplicity can be expressed as

$$\langle n\rangle = \frac{1}{p} = C\,\mathrm{e}^{\alpha y} \qquad \Rightarrow \qquad y = \frac{1}{\alpha}\ln\left(\frac{\langle n\rangle}{C}\right), \tag{4}$$

and, by applying Eq. (1), a relation between the mean multiplicity and the entropy [20] can be obtained as

$$S(y) = \langle n\rangle \ln\langle n\rangle - (\langle n\rangle - 1)\ln(\langle n\rangle - 1) \approx \ln(\langle n\rangle) + 1\,, \tag{5}$$

where, in the last step, we took the $C = 1$ in the sub-asymptotic limit [4].

2.2. Generalized Mueller model

A new cascade equation that was introduced in Ref. [21] has a new parameter

$$\partial_y P_n(y) = -\alpha(n+2h)P_n(y) + \alpha(n-1+2h)P_{n-1}(y)\,, \tag{6}$$

and its solution has the form of the negative binomial distribution with the above defined $p = \frac{1}{C}\,\mathrm{e}^{-\alpha y}$ probability

$$P_n(y) = \frac{\Gamma(2h+n)}{n!\Gamma(2h)} p^{2h}\,(1-p)^n\,. \tag{7}$$

The equation has been obtained in the context of study of the Krylov complexity [24–26] for the coherent states of the conformal SL(2,R) group and mapped to QCD in Ref. [21]. Here, h is a conformal weight of this group which tells how the operators of the theory scale. As an example, in the previous section, we described the Mueller model that has the conformal weight $h = 1/2$.

In our studies, it also measures the deviation from the Mueller case and the dispersion of the NBD with $k_{\mathrm{NBD}} = 2h$. It is also worth emphasizing that we utilize a version of the model that allows for p_0, which effectively corresponds to the vacuum contribution. Such contributions were accounted for in recent studies [27, 28] and successful description of hadronic entropy data were obtained.

An explicit form of the entropy of this distribution is not known (an implicit, closed expression is given in Ref. [29]). The mean multiplicity of dipoles or, in the context of Ref. [21], the Krylov complexity (which is the measure of the spread of the underlying quantum state in Hilbert space), as a function of rapidity, is given in the following form:

$$\langle n\rangle = 2h\,(C\mathrm{e}^{\alpha y} - 1)\,. \tag{8}$$

2.3. Comparison of the models

We show the comparison of the two studied models in terms of logarithm of average multiplicity and entropy. The models are evaluated with parameters based on fits that will be discussed in Section 3. From the solutions given in Eqs. (3) and (7), the multiplicity distribution for the original dipole model features exponential decay, while the distributions from the generalized model are more dispersed and closely resemble the measured multiplicity data in *pp* [8, 10–15, 17] and *ep* [1]. It can be seen in Fig. 1 that the generalized model predicts higher average number of partons and higher entropy in the investigated rapidity range.

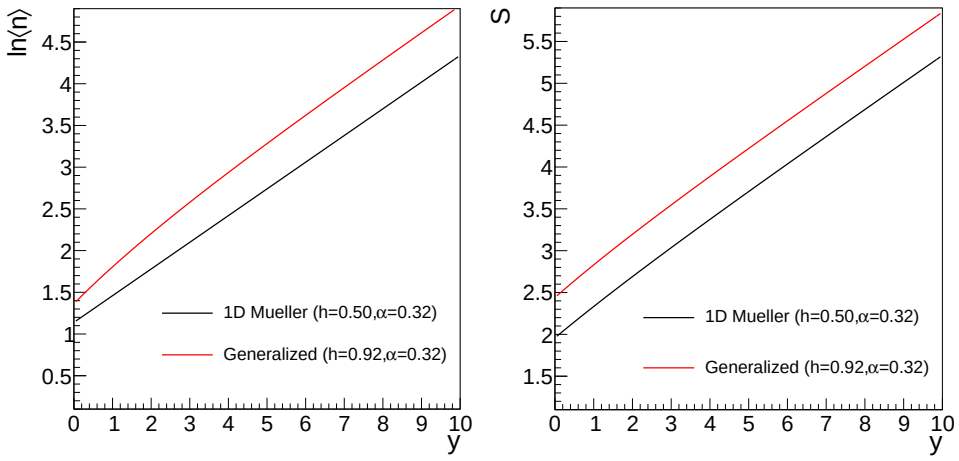


Fig. 1. The log average multiplicity (left) and the entropy (right) as a function of the rapidity calculated from the two models.

3. Comparison to data

We calculated $S(\ln\langle n\rangle)$ for both models and compared the results with data from *pp* collision at energies ranging from $\sqrt{s_{NN}} = 200$ GeV to 13 TeV, and for pseudorapidity $|\eta| < 0.5$ to $4.5 < \eta < 5$ [8, 10–15, 17]. We found that the generalized model can describe well the data, however, the Mueller model exhibits deviation from the data at lower multiplicities as it can be seen in Fig. 2. The obtained model parameters and the goodness of the fits can be seen in Table 1.

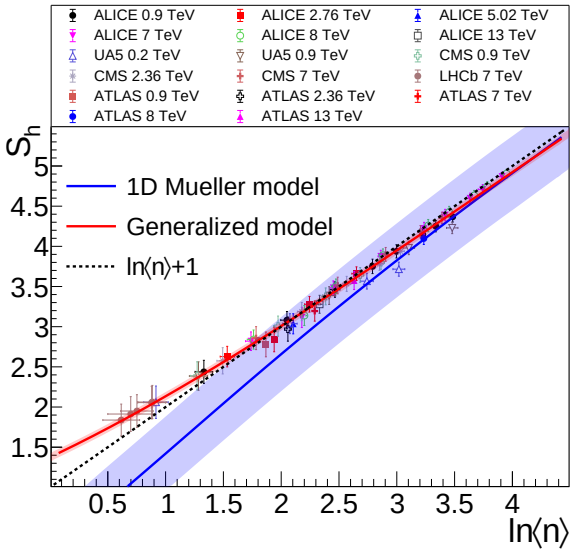


Fig. 2. The new observable compared to data

Table 1. The parameters and fit qualities of the dipole models [18–21]. The fixed values are based on earlier results [27].

Parameter	1D dipole	Generalized model
h	0.50 (fixed)	0.92 ± 0.05
α	0.32 (fixed)	0.32 (fixed)
C	3.13 ± 0.48	3.13 (fixed)
χ^2/NDF	309/63	24/63

4. Summary and outlook

We described the 1D Mueller dipole model and its generalized version. We introduced a new observable, $S(\ln\langle n\rangle)$, that has been compared to *pp* data spanning 2 orders of magnitude in energy and covers a large pseudorapidity interval. We found that the generalized model can describe the data, while the Mueller model deviates from the data points at lower multiplicities.

We would like to emphasize that such studies could be valuable in the context of saturation if the model were extended to take into account recombination effects. Also, theoretical calculations could improve the description of *pp* data, which represent a more complex colliding environment than DIS processes. Experimentally, the methodological issue addressed could be resolved and further, more detailed analyses could refine the present and future results.

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